CHAPTER 6

Rational Choice under Risk and Uncertainty

6.1 Introduction

In Part 2, we left the theory of decision aside for a moment in order to talk about judgment. Now it is time to return to questions of decision, and specifically, rational decision. In this chapter we explore the theory of rational choice under risk and uncertainty. According to the traditional perspective, you face a **choice under uncertainty** when the probabilities of the relevant outcomes are completely unknown or not even meaningful; you face a **choice under risk** when the probabilities of the relevant outcomes are both meaningful and known. At the end of the day, we want a theory that gives us principled answers to the question of what choice to make in any given decision problem. It will take a moment to develop this theory. We begin by discussing uncertainty and then proceed to expected value, before getting to expected utility. Ultimately, expected-utility theory combines the concept of utility from Chapter 2 with the concept of probability from Chapter 4 into an elegant and powerful theory of choice under risk.

6.2 Uncertainty

Imagine that you are about to leave your house and have to decide whether to take an umbrella or to leave it at home (Figure 6.1). You are concerned that it might rain. If you do not take the umbrella and it does not rain, you will spend the day dry and happy; if you do not take the umbrella and it does rain, however, you will be wet and miserable. If you take the umbrella, you will be dry no matter, but carrying the cumbersome umbrella will infringe on your happiness. Your decision problem can be represented as in Table 6.1(a).

In a table like this, the left-most column represents your menu, that is, the options available to you. Other than that, there is one column for each thing that can happen. These things are referred to as **states of the world** or simply **states**, listed in the top row. Together, they constitute the outcome space. In this case, obviously, there are only two states: either it rains, or it does not. Nothing prevents expressing your preferences over the four outcomes by using our old friend the utility function from Section 2.7. Utility payoffs can be represented as in Table 6.1(b). Under the circumstances, what is the rational thing to do? Let us assume that you treat this as a choice under uncertainty. There are a number of different criteria that could be applied.

According to the **maximin criterion**, you should choose the alternative that has the greatest minimum utility payoff. If you take the umbrella, the minimum

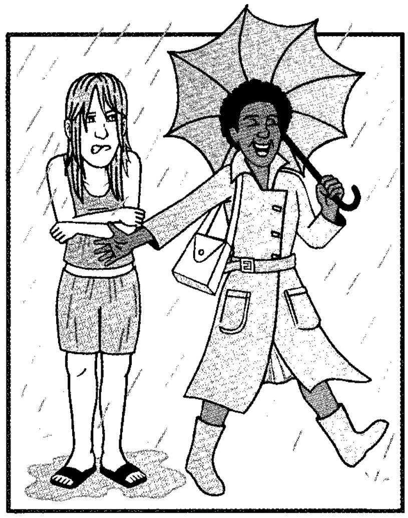

Figure 6.1 Ready for rain? Illustration by Cody Taylor

payoff is three; if you leave the umbrella at home, the minimum payoff is zero. Consequently, maximin reasoning would favor taking the umbrella. According to the **maximax criterion**, you should choose the alternative that has the greatest maximum utility payoff. If you take the umbrella, the maximum payoff is three; if you leave the umbrella at home, the maximum payoff is five. Thus, maximax reasoning would favor leaving the umbrella at home. The maximin reasoner is as cautious as the maximax reasoner is reckless. The former looks at nothing but the *worst* possible outcome associated with each act, and the latter looks at nothing but the *best* possible outcome associated with each act. Some of my students started referring to the maximax criterion as the **YOLO criterion** – from "You Only Live Once," for the reader unfamiliar with millennial.

Table 6.1 Umbrella problem

	Rain	No rain
Take umbrella	Dry, not happy	Dry, not happy
Leave umbrella	Wet, miserable	Dry, happy

(a) Payoffs

	Rain	No rain
Take umbrella	3	3
Leave umbrella	0	5

(b) Utility payoffs

	Rain	No rain
Take umbrella	0	2
Leave umbrella	3	0

(c) Risk payoffs

Exercise 6.1 The watch Having just bought a brand new watch, you are asked if you also want the optional life-time warranty.

(a) Would a maximin reasoner purchase the warranty?

(b) What about a maximax reasoner?

There are other criteria as well. According to the **minimax-risk criterion**, you should choose the alternative that is associated with the lowest maximum **risk** or **regret**. If you take the umbrella and it rains, or if you leave the umbrella at home and it does not rain, you have zero regrets. If you take the umbrella and it does not rain, your regret equals the best payoff you could have had if you had acted differently (five) minus your actual payoff (three), that is, two. By the same token, if you leave the umbrella at home and it does rain, your regret equals three. These "risk payoffs" can be captured in table form, as shown in Table 6.1(c). Since bringing the umbrella is associated with the lowest maximum regret (two, as opposed to three), minimax-risk reasoning favors taking the umbrella. The term **regret aversion** is sometimes used when discussing people's tendency to behave in such a way as to minimize anticipated regret. Regret aversion may be driven by loss aversion (see Section 3.5), since regret is due to the loss of a payoff that could have resulted from the state that obtains, had the agent acted differently. (We return to the topic of regret in Section 7.4.)

Exercise 6.2 Rational choice under uncertainty This exercise refers to the utility matrix of Table 6.2. What course of action would be favored by (**a**) the maximin criterion, (**b**) the maximax criterion, and (**c**) the minimax-risk criterion? As part of your answer to (c), make sure to produce the risk-payoff matrix.

Table 6.2 Decision under uncertainty

	S_1	S_2
A	1	10
B	2	9
C	3	6

Quite a number of authors have offered advice about how to minimize regret. In *The Picture of Dorian Gray*, Oscar Wilde wrote: "Nowadays most people die of a sort of creeping common sense, and discover when it is too late that the only things one never regrets are one's mistakes." And books with titles such as *The Top Five Regrets of the Dying* aspire to tell us what we too might one day come to regret: working too much, trying to please others, not allowing ourselves to be happy, and so on. Then again, Søren Kierkegaard – sometimes called "the father of existentialism" – altogether despaired of avoiding regret. "My honest opinion and my friendly advice is this: Do it or do not do it – you will regret both," he wrote in a book titled *Either/Or*.

Problem 6.3 The dating game under uncertainty *Imagine that you are considering whether or not to ask somebody out on a date.*

(a) Given your utility function, what course of action would be favored by (i) the maximin criterion, (ii) the maximax criterion, and (iii) the minimax-risk criterion?

(b) In the words of Alfred, Lord Tennyson, "'Tis better to have loved and lost / Than never to have loved at all." What decision criterion do these lines advocate?

Of all criteria for choice under uncertainty, the maximin criterion is the most prominent. It is, among other things, an important part of the philosopher John

Rawls's theory of justice. In Rawls's theory, the principles of justice are the terms of cooperation that rational people would agree to follow in their interactions with each other if they found themselves behind a "veil of ignorance," meaning that they were deprived of all morally relevant information about themselves, the society in which they live, and their place in that society. Suppose, for example, that you have to choose whether to live either in a society with masters and slaves or in a more egalitarian society, without knowing whether (in the former) you would be master or slave. According to Rawls, the rational procedure is to rank societies in accordance with the worst possible outcome (for you) in each society – that is, to apply the maximin criterion – and to choose the more egalitarian option. Rawls took this to constitute a reason to think that an egalitarian society is more just than a society of masters and slaves.

Critics have objected to the use of maximin reasoning in this and other scenarios. One objection is that maximin reasoning fails to consider relevant utility information, since for each act, it ignores all payoffs except the worst. Consider the two decision problems in Table 6.3. Maximin reasoning would favor A in either scenario. Yet it does not seem completely irrational to favor A over B and B* over A, since B* but not B upholds the prospect of ten billion utiles.

Table 6.3 More decisions under uncertainty

	S_1	S_2
A	1	1
B	0	10

(a)

	S_1	S_2
A	1	1
B*	0	10^{10}

(b)

Another objection is that maximin reasoning fails to take into account the chances that the various states of the world will obtain. In a famous critique of Rawls's argument, Nobel prize-winning economist John C. Harsanyi offered the following example:

Example 6.4 Harsanyi's challenge Suppose you live in New York City and are offered two jobs at the same time. One is a tedious and badly paid job in New York City itself, while the other is a very interesting and well-paid job in Chicago. But the catch is that, if you wanted the Chicago job, you would have to take a plane from New York to Chicago (for example, because this job would have to be taken up the very next day). Therefore, there would be a very small but positive probability that you might be killed in a plane accident.

Assuming that dying in a plane crash is worse than anything that could happen on the streets of New York, as Harsanyi points out, maximin reasoning would favor the tedious NYC job, *no matter* how much you prefer the Chicago job and *no matter* how unlikely you think a plane accident might be. This does not sound quite right.

Perhaps there are scenarios in which the probabilities of the relevant outcomes are completely unknown or not even meaningful, and perhaps in those scenarios maximin reasoning – or one of the other criteria discussed earlier

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in this section – is appropriate. Yet the upshot is that, whenever possible, it is perfectly reasonable to pay attention to all possible payoffs as well as to the probabilities that the various states might obtain. When facing the umbrella problem in Table 6.1, for example, it seems rational to take into account all four cells in the payoff matrix as well as the probability of rain.

6.3 **Expected value**

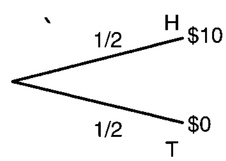
From now on I will assume that it is both meaningful and possible to assign probabilities to outcomes; that is, we will be leaving the realm of choice under uncertainty and entering the kingdom of choice under risk. In this section, we explore one particularly straightforward approach – expected value – that takes the entire payoff matrix as well as probabilities into account.

The **expected value** of a gamble is what you can expect to win *on the average, in the long run,* when you play the gamble. Suppose I make you the following offer: I will flip a fair coin, and I will give you \$10 if the coin comes up heads (H), and nothing if the coin comes up tails (T). This is a reasonably good deal: with 50 percent probability you will become \$10 richer. This gamble can easily be represented in tree and table form, as shown in Figure 6.2. It is clear that on the average, in the long run, you would get \$5 when playing this gamble; in other words, the expected value of the gamble is \$5. That is the same as the figure you get if you multiply the probability of winning (1/2) by the dollar amount you stand to win (\$10).

Exercise 6.5 Lotto 6/49 Represent the gamble accepted by someone who plays Lotto 6/49 (from Exercise 4.28 on page 85) as in Figure 6.2(a) and (b). Assume that the grand prize is a million dollars.

Example 6.6 Lotto 6/49, cont. What is the expected value of a Lotto 6/49 ticket, if the grand prize is a million dollars?

We know from Exercise 4.28 that the ticket is a winner one time out of 13,983,816. The means that the ticket holder will receive, on the average, in the



(a)

	H	T
Offer	\$10	\$0

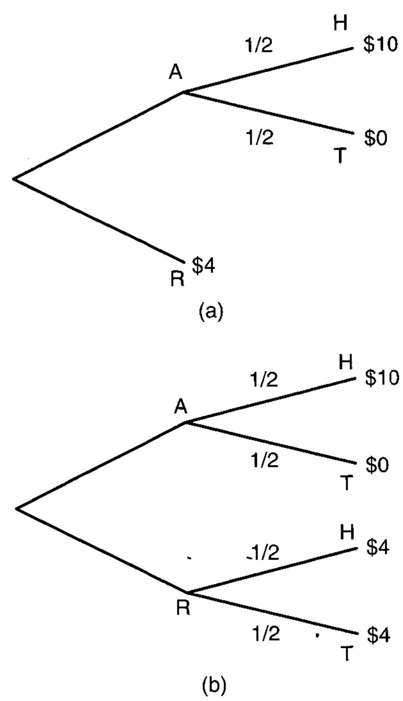
(b)

Figure 6.2 Simple gamble

long run, 1/13,983,816 * $1,000,000. You get the same answer if you multiply the probability of winning by the amount won: 0.000,000,07 * $1,000,000 = $0.07. That is 7 cents.

Problem 6.7 *What would you pay to play this gamble? If you are willing to pay to play this game, what do you hope to achieve?*

Sometimes two or more acts are available to you, in which case you have a choice to make. Imagine, for instance, that you can choose between the gamble in Figure 6.2 and $4 for sure. If so, we can represent your decision problem in tree form as shown in Figure 6.3(a). We can also think of the outcome of rejecting the gamble as $4 no matter whether the coin comes up heads or tails. Thus, we can think of the gamble as identical to that in Figure 6.3(b). The latter decision tree makes it obvious how to represent this gamble in table form (see Figure 6.3(c)). The numbers in the row marked "Reject" represent the fact that if you reject the gamble, you keep the four dollars whether or not it turns out to be a winner.



	H	T
Accept	$10	$0
Reject	$4	$4

(c)

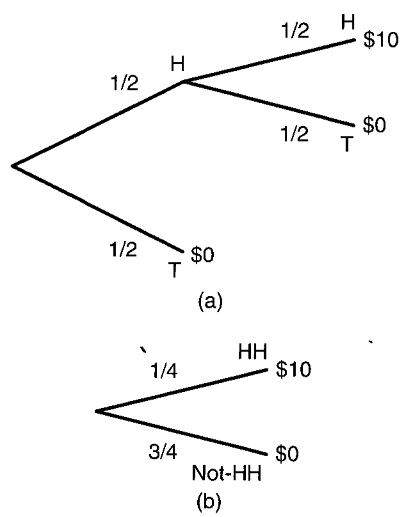
Figure 6.3 Choice between gambles

Exercise 6.8 Expected value For the following questions, refer to Figure 6.3(c).
(a) What is the expected value of accepting this gamble?
(b) What is the expected value of rejecting it?

You may be wondering if all gambles can be represented in table form: they can. Consider, for instance, what would happen if you win the right to play the gamble from Figure 6.2 if you flip a coin and it comes up heads. If so, the complex gamble you are playing would look like Figure 6.4(a). The key to analyzing more complex, multi-stage gambles like this is to use one of the AND rules to construct a simpler one. In this case, the gamble gives you a 1/4 probability of winning \$10 and a 3/4 probability of winning nothing. Hence, the gamble can also be represented as in Figure 6.4(b). This makes it obvious how to represent the complex gamble in table form, as in Figure 6.4(c). There can be more than two acts or more than two states. So, in general, we end up with a matrix like Table 6.4. By now, it is obvious how to define expected value.

Definition 6.9 Expected value *Given a decision problem as in Table 6.4, the expected value $EV(A_i)$ of an act A_i is given by:*

$$EV(A_i) = \Pr(S_1) * C_{i1} + \Pr(S_2) * C_{i2} + \ldots + \Pr(S_n) * C_{in}$$
$$= \sum_{j=1}^{n} \Pr(S_j) C_{ij}.$$



	HH	¬HH
Gamble	\$10	\$0

(c)

Figure 6.4 Multi-stage gamble

If this equation looks complicated, notice that actual computations are easy. For each state – that is, each column in the table – you multiply the probability of that state occurring with what you would get if it did; then you add all your numbers up. If you want to compare two or more acts, just complete the procedure for each act and compare your numbers. As you can tell, this formula gives some weight to each cell in the payoff matrix and to the probabilities that the various states of the world will obtain.

Table 6.4 The general decision problem

	S_1	S_2	...	S_n
A_1	C_{11}	C_{12}	...	C_{1n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	C_{m1}	C_{m2}	...	C_{mn}

The table form is often convenient when computing expected values, since it makes it obvious how to apply the formula in Definition 6.9. The fact that more complex gambles can also be represented in table form means that the formula applies even in the case of more complex gambles. Hence, at least as long as outcomes can be described in terms of dollars, lives lost, or the like, the concept is well defined. To illustrate how useful this kind of knowledge can be, let us examine decision problem that you might encounter in casinos and other real-world environments.

Exercise 6.10 Roulette A roulette wheel has slots numbered 0, 00, 1, 2, 3, ..., 36 (see Figure 6.5). The players make their bets, the croupier spins the wheel, and depending on the outcome, payouts may or may not be made. Players can make a variety of bets. Table 6.5 lists the bets that can be made as well as the associated payoffs for a player who wins after placing a one-dollar bet. Fill in the table.

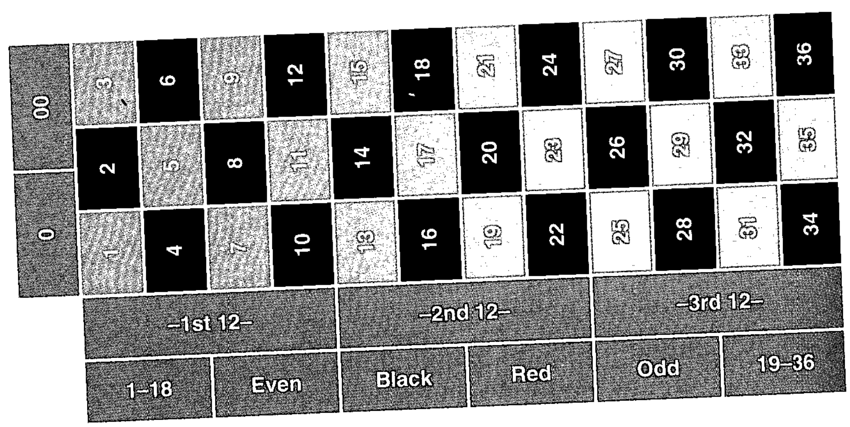


Figure 6.5 Roulette table

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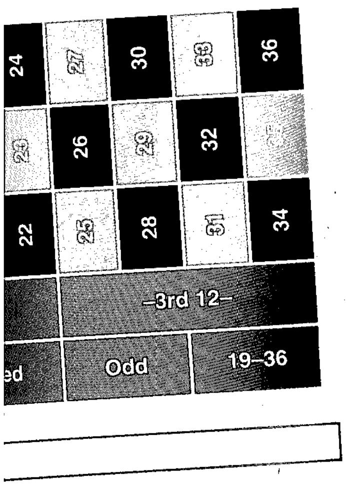

Table 6.5 Roulette bets

Bet	Description	Payout	Pr(win)	Expected Value
Straight Up	One number	\$36		
Split	Two numbers	\$18		
Street	Three numbers	\$12		
Corner	Four numbers	\$9		
First Five	0, 00, 1, 2, 3	\$7		
Sixline	Six numbers	\$6		
First 12	1–12	\$3		
Second 12	13–24	\$3		
Third 12	25–36	\$3		
Red		\$2		
Black		\$2		
Even		\$2		
Odd		\$2		
Low	1–18	\$2		
High	19–36	\$2		

Exercise 6.11 Parking You are considering whether to park legally or illegally and decide to be rational about it. Use negative numbers to represent costs in your expected-value calculations.

(a) Suppose that a parking ticket costs \$30 and that the probability of getting a ticket if you park illegally is $1/5$. What is the expected value of parking illegally?

(b) Assuming that you use expected-value calculations as a guide in life, would it be worth paying \$5 in order to park legally?

It is perfectly possible to compute expected values when there is more than one state with a non-zero payoff.

Example 6.12 You are offered the following gamble: if a (fair) coin comes up heads, you receive \$10; if the coin comes up tails, you pay \$10. What is the expected value of this gamble?

The expected value of this gamble is $1/2 * 10 + 1/2 * (-10) = 0$.

Exercise 6.13 Suppose somebody intends to roll a fair die and pay you \$1 if she rolls a one, \$2 if she rolls a two, and so on. What is the expected value of this gamble?

Exercise 6.14 ***Deal or No Deal*** You are on the show *Deal or No Deal*, where you are facing so many boxes, each of which contains some (unknown) amount of money (see Figure 6.6). At this stage, you are facing three boxes. One of them contains $900,000, one contains $300,000, and one contains $60, but you do not know which is which. Here are the rules: if you choose to open the boxes, you can open them in any order you like, but you can keep the amount contained in the *last* box only.

(a) What is the expected value of opening the three boxes?
(b) The host gives you the choice between a sure $400,000 and the right to open the three boxes. Assuming you want to maximize expected value, which should you choose?
(c) You decline the $400,000 and open a box. Unfortunately, it contains the $900,000. What is the expected value of opening the remaining two boxes?
(d) The host gives you the choice between a sure $155,000 and the right to open the remaining two boxes. Assuming you want to maximize expected value, which should you choose?

Expected-value calculations form the core of **cost–benefit analysis**, which is used to determine whether all sorts of projects are worth undertaking. Corporations engage in cost–benefit analysis to determine whether to invest in a new plant, start a new marketing campaign, and so on. Governments engage in cost–benefit analysis to determine whether to build bridges, railways, and airports; whether to incentivize foreign corporations to relocate there; whether to overhaul the tax system; and many other things. The basic idea is simply to compare expected benefits with expected costs: if the benefits exceed or equal the cost, the assumption is that it is worth proceeding, otherwise not.

So far, we have used our knowledge of probabilities to compute expected values. It is also possible to use Definition 6.9 to compute probabilities, provided that we know enough about the expected values. So, for example, we can ask the following question.

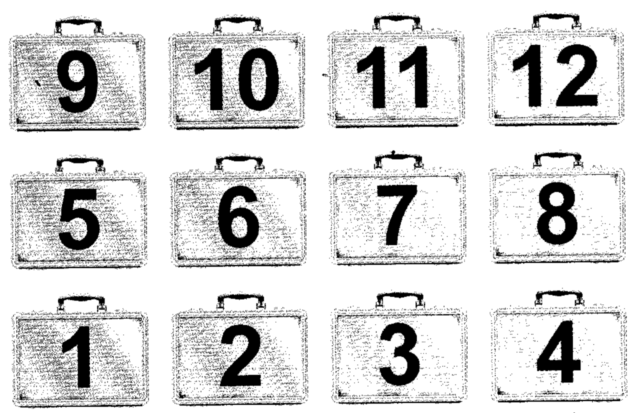

Figure 6.6 *Deal or No Deal*

Example 6.15 Parking, cont. If a parking ticket costs \$30, and it costs \$5 to park legally, what does the probability of getting a ticket need to be for the expected value of parking legally to equal the expected value of parking illegally?

We solve this problem by setting up an equation. Assume, first of all, that the probability of getting a parking ticket when you park illegally is p. Assume, further, that the expected value of parking illegally equals the expected value of parking legally: $p * (-30) = -5$. Solving for p, we get that $p = -5/-30 = 1/6$. This means that if the probability of getting a ticket is $1/6$, the expected values are identical. If p is greater than $1/6$, the expected value of parking legally is greater than the expected value of parking illegally; if p is lower than $1/6$, the expected value of parking legally is smaller than the expected value of parking illegally.

Exercise 6.16 Parking, cont. Assume that the cost of parking legally is still \$5.

(a) If the parking ticket costs \$100, what does the probability need to be for the expected value of parking legally to equal the expected value of parking illegally?

(b) What if the ticket costs \$10?

Problem 6.17 Parking, cont. *Given what you pay for parking and given what parking fines are in your area, what does the probability of getting a ticket need to be for the expected value of parking legally to equal the expected value of parking illegally?*

There is a whole field called **law and economics** that addresses questions such as this, exploring the conditions under which it makes sense for people to break the law, and how to design the law so as to generate the optimal level of crime.

Exercise 6.18 Lotto 6/49, cont. Suppose a Lotto 6/49 ticket costs \$1 and that the winner will receive \$1,000,000. What does the probability of winning need to be for this lottery to be **actuarially fair**, that is, for its price to equal its expected value?

Exercise 6.19 Warranties A tablet computer costs \$325; the optional one-year warranty, which will replace the tablet computer at no cost if it breaks, costs \$79. What does the probability p of the tablet computer breaking need to be for the expected value of purchasing the optional warranty to equal the expected value of not purchasing it?

As this example suggests, the price of warranties is often inflated relative to the probability that the product will break (for the average person, anyway).

Unfortunately, when used as a guide in life, expected-value calculations have drawbacks. Obviously, we can only compute expected values when consequences can be described in terms of dollars, lives lost, or similar. The definition of expected value makes no sense if the consequences C_{ij} are not expressed in numbers. Moreover, under many conditions expected-value considerations give apparently perverse advice, and therefore cannot serve as a general guide to decision-making in real life. Consider, for example, what to do if you have 30 minutes in a casino before the mafia comes after you to reclaim your debts. Assuming that you will be in deep trouble unless you come up with, say, $10,000 before they show up, gambling can be a very reasonable thing to do, even if the expected values are low. Consider, finally, the following famous example.

Example 6.20 St Petersburg paradox A gamble is resolved by tossing an unbiased coin as many times as necessary to obtain heads. If it takes only one toss, the payoff is $2; if it takes two tosses, it is $4; if it takes three, it is $8; and so forth (see Table 6.6). What is the expected value of the gamble?

Notice that the probability of getting heads on the first flip (H) is 1/2; the probability of getting tails on the first flip and heads on the second (TH) is 1/4; the probability of getting tails on the first two flips and heads on the third (TTH) is 1/8; and so on. Thus, the expected value of the gamble is:

$$\frac{1}{2} * \$2 + \frac{1}{4} * \$4 + \frac{1}{8} * \$8 + \ldots =$$
$$\$1 \quad + \quad \$1 \quad + \quad \$1 \quad + \ldots = \$\infty.$$

In sum, the expected value of the gamble is *infinite*. This means that if you try to maximize expected value, you should be willing to pay any (finite) price for this gamble. That does not seem right, which is why the result is called the **St Petersburg paradox**. And it would not seem right even if you could trust that you would receive the promised payoffs no matter what happens.

Table 6.6 St Petersburg gamble

	H	TH	TTH	...
St Petersburg gamble	$2	$4	$8	...

6.4 Expected utility

Our calculations in Section 4.4 suggested that games like Lotto 6/49 are simply not worth playing (see Exercise 4.28). But the story does not end there. For one thing, as our deliberations in this chapter have illustrated, the size of the prize matters (see, for instance, Example 6.6). Equally importantly, a dollar is not as valuable as every other dollar. You may care more about a dollar bill if it is the first in your pocket than if it is the tenth. Or, if the mafia is coming after you to settle a $10,000 debt, the first 9999 dollar bills may be completely useless to you, since you will be dead either way, whereas the 10,000th can save your life.

To capture this kind of phenomenon, and to resolve the St Petersburg paradox, we simply reintroduce the concept of utility from Section 2.7. The utility of money is often represented in a graph, with money (or wealth, or income)

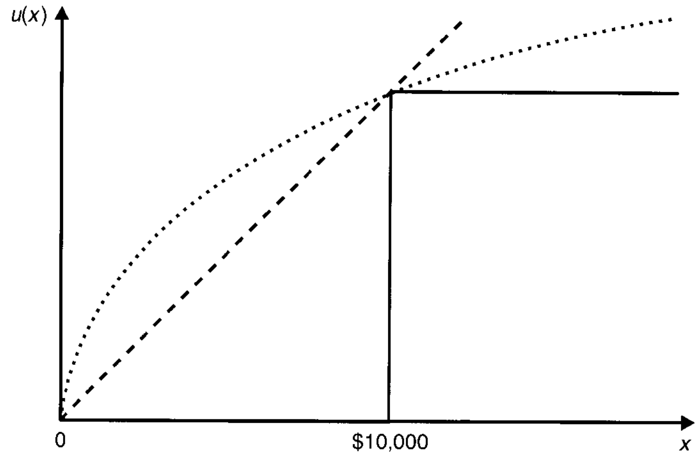

Figure 6.7 The utility of money

on the x-axis and utility on the y-axis. In Figure 6.7, for example, the dashed line represents the expected value of money if $u(x) = x$; the solid line represents the utility of money if the mafia is coming after you; and the dotted line represents the case when a dollar becomes worth less and less as you get more of them, that is, when the marginal utility of money is diminishing. When the curve bends downwards when you move to the right, as the dotted line does, it is said to be **concave**.

For most goods, it is probably true that the marginal utility is diminishing. When buying a newspaper from a box, you put a few coins in the slot, open a door, and grab a newspaper from the stack. Nothing prevents you from grabbing two or more copies, but most people do not. Why? Newspaper boxes work because the marginal utility of newspapers is sharply diminishing. While the first copy of the *Wall Street Journal* permits you to learn what is going on in the markets, subsequent copies are best used to make funny hats or to wrap fish. There are exceptions to the rule, however. Beer is not sold like newspapers, and for good reason: its marginal utility is not diminishing and may even be increasing. As you may have read in a book, after people have had a beer, a second beer frequently seems like an even better idea than the first one did, and so forth. This is why a beer – as in "Let's have a beer" – is a mythical animal not unlike unicorns.

How does this help? Consider the St Petersburg gamble from Section 6.3. Let us assume that the marginal utility for money is diminishing, which seems plausible.

Table 6.7 St Petersburg gamble with utilities

	H	TH	TTH	…
St Petersburg gamble	log(2)	log(4)	log(8)	…

Mathematically, the utility of a given amount of money x might equal the logarithm of x, so that $u(x) = \log(x)$. If so, we can transform Table 6.6 into a table in which consequences are expressed in utilities instead of dollars (see Table 6.7). Now, we can compute the **expected utility** of the gamble. The expected utility of a gamble is the amount of utility you can expect to gain *on the average, in the long run*, when you play the gamble. In the case of the St Petersburg gamble, the expected utility is:

$$\frac{1}{2} * \log(2) + \frac{1}{4} * \log(4) + \frac{1}{8} * \log(8) + \ldots \approx 0.602 < \infty.$$

This way, the expected utility of the St Petersburg gamble is well-defined and finite. (In Example 6.37 we will compute what the gamble is worth in dollars and cents.) Formally, this is how we define expected utility:

Definition 6.21 Expected utility *Given a decision problem like Table 6.4, the expected utility $EU(A_i)$ of an act A_i is given by:*

$$\begin{aligned} EU(A_i) &= \Pr(S_1) * u(C_{i1}) + \Pr(S_2) * u(C_{i2}) + \ldots + \Pr(S_n) * u(C_{in}) \\ &= \sum_{j=1}^{n} \Pr(S_j) u(C_{ij}) \end{aligned}$$

Somebody who chooses that option with the greatest expected utility is said to engage in **expected-utility maximization**. Examples like the St Petersburg paradox suggest that expected-utility maximization is both a better guide to behavior, and a better description of actual behavior, than expected-value maximization. That is, the theory of expected utility is a better normative theory, and a better descriptive theory, than the theory of expected value. Computing expected utilities is not much harder than computing expected values, except that you need to multiply each probability with the utility of each outcome.

Example 6.22 Expected utility Consider, again, the gamble from Figure 6.3(c). Suppose that your utility function is $u(x) = \sqrt{x}$. Should you accept or reject the gamble?

The utility of rejecting the gamble is $EU(\text{R}) = u(4) = \sqrt{4} = 2$. The utility of accepting the gamble is $EU(\text{A}) = 1/2 * u(10) + 1/2 * u(0) = 1/2 * \sqrt{10} \approx 1.58$. The rational thing to do is to reject the gamble.

Exercise 6.23 Expected utility, cont. Suppose instead that your utility function is $u(x) = x^2$.

(a) What is the expected utility of rejecting the gamble?
(b) What is the expected utility of accepting the gamble?
(c) What should you do?

When the curve bends upwards as you move from left to right, like the utility function $u(x) = x^2$ does, the curve is said to be **convex**.

Exercise 6.24 Lotto 6/49, cont. Assume still that your utility function is $u(x) = \sqrt{x}$, that the probability of winning at Lotto 6/49 is one in 13,983,816, and that the prize is a million dollars.

(a) What is the expected utility of holding a Lotto ticket?

(b) What is the expected utility of the dollar you would have to give up in order to receive the Lotto ticket?

(c) Which would you prefer?

Exercise 6.25 Expected utility, again Suppose that you are facing three gambles. A gives you a 1/3 probability of winning \$9. B gives you a 1/4 probability of winning \$16. C gives you a 1/5 probability of winning \$25.

(a) What is the expected utility of each of these gambles if your utility function is $u(x) = \sqrt{x}$, and which one should you choose?

(b) What is the expected utility of each of these gambles if your utility function is $u(x) = x^2$, and which one should you choose?

Exercise 6.26 Expected value and expected utility Assume again that your utility function is $u(x) = \sqrt{x}$. Compute (i) the expected value and (ii) the expected utility of the following gambles:

(a) G: You have a 1/4 chance of winning \$25 and a 3/4 chance of winning \$1.

(b) G*: You have a 2/3 chance of winning \$7 and a 1/3 chance of winning \$4.

Another major advantage of the expected-utility framework is that it can be applied to decisions that do not involve consequences expressed in terms of dollars, lives lost, or the like. The expected-utility formula can be used quite generally, as long as it is possible to assign utilities to all outcomes. That is to say that expected utilities can be calculated whenever you have preferences over outcomes – which you do, if you are rational. Hence, expected-utility theory applies, at least potentially, to all decisions. The following exercises illustrate how expected-utility reasoning applies even when consequences are not obviously quantifiable.

Exercise 6.27 Hearing loss A patient with hearing loss is considering whether to have surgery. If she does not have the surgery, her hearing will get no better and no worse. If she does have the surgery, there is an 85 percent chance that her hearing will improve, and a five percent chance that it will deteriorate. If she does not have the surgery, her utility will be zero. If she does have the surgery and her hearing improves, her utility will be ten. If she does have the surgery but her hearing is no better and no worse, her utility will be minus two. If she does have the surgery and her hearing deteriorates, her utility will be minus ten.

(a) Draw a tree representing the decision problem.

(b) Draw a table representing the problem.

(c) What is the expected utility of not having the operation?

(d) What is the expected utility of having the operation?

(e) What should the patient do?

Exercise 6.28 Thanksgiving indecision Suppose you are contemplating whether to go home for Thanksgiving. You would like to see your family, but you are worried that your aunt may be there, and you genuinely hate your aunt. If you stay in town you are hoping to stay with your roommate, but then again, there is some chance that she will leave town. The probability that your aunt shows up is 1/4, and the probability that your roommate leaves town is 1/3. The utility of celebrating Thanksgiving with your family without the aunt is 12 and with the aunt is minus two. The utility of staying in your dorm without your roommate is three and with the roommate is nine.

(a) Draw a decision tree.
(b) Calculate the expected utility of going home and of staying in town.
(c) What should you do?

Exercise 6.29 Pascal's wager The French seventeenth-century mathematician and philosopher Blaise Pascal suggested the following argument for a belief in God. The argument is frequently referred to as **Pascal's wager**. Either God exists (G), or He does not (¬G). We have the choice between believing (B) or not believing (¬B). If God does not exist, it does not matter if we believe or not: the utility would be the same. If God does exist, however, it matters a great deal: if we do believe, we will enjoy eternal bliss; if we do not believe, we will burn in hell.

(a) Represent the decision problem in table form, making up suitable utilities as you go along.
(b) Let p denote the probability that G obtains. What is the expected utility of B and ¬B?
(c) What should you do?

Notice that it does not matter what p is. B **dominates** ¬B in the sense that B is associated with a higher expected utility no matter what.

Of course, Definition 6.21 can also be used to compute probabilities provided that we know enough about the expected utilities. So, for example, we can ask the following kinds of question.

Example 6.30 Umbrella problem, cont. This question refers to Table 6.1(b), that is, the umbrella problem from Section 6.2. If the probability of rain is p, what does p need to be for the expected utility of taking the umbrella to equal the expected utility of leaving it at home?

To answer this problem, set up the following equation: *EU* (Take umbrella) = *EU* (Leave umbrella). Given the utilities in Table 6.1(b), this implies that $3 = p * 0 + (1 - p) * 5$, which implies that $p = 2/5$.

Exercise 6.31 Indifference This question refers to Table 6.2. Let p denote the probability that S_1 obtains.

(a) If an expected-utility maximizer is indifferent between A and B, what is his p?
(b) If another expected-utility maximizer is indifferent between B and C, what is her p?
(c) If a third expected-utility maximizer is indifferent between A and C, what is their p?

6.5 Attitudes toward risk

As you may have noticed already, expected-utility theory has implications for attitudes toward risk. Whether you reject a gamble (as in Example 6.22) or accept it (as in Exercise 6.23) depends, at least to some extent, on the shape of your utility function. This means that we can explain people's attitudes toward risk in terms of the character of their utility function.

The theory of expected utility can explain why people often reject a gamble in favor of a sure dollar amount equal to its expected value. We simply have to add to the theory of expected utility the auxiliary assumption of diminishing marginal utility of money.

Example 6.32 Risk aversion Suppose you own \$2 and are offered a gamble giving you a 50 percent chance of winning a dollar and a 50 percent chance of losing a dollar. This decision problem can be represented as in Figure 6.8. Your utility function is $u(x) = \sqrt{x}$, so that marginal utility is diminishing. Should you take the gamble?

The problem can be represented as in Table 6.8. Expected-utility calculations show that you should reject the gamble, since $EU(\text{Accept}) = 1/2 * \sqrt{3} + 1/2 * \sqrt{1} \approx 1.37$ and $EU(\text{Reject}) = \sqrt{2} \approx 1.41$.

Table 6.8 Another gambling problem

	Win (1/2)	Lose (1/2)
Accept (A)	$\sqrt{3}$	$\sqrt{1}$
Reject (R)	$\sqrt{2}$	$\sqrt{2}$

An expected-value maximizer would have been indifferent between accepting and rejecting this gamble, since both expected values are \$2. Trivially, then, if your utility function is $u(x) = x$, you will be indifferent between the two options. For comparison, consider the following problem.

Exercise 6.33 Risk proneness Consider, again, the gamble in Figure 6.8. Now suppose that your utility function is $u(x) = x^2$. Unlike the previous utility function, which gets flatter when amounts increase, this utility function gets

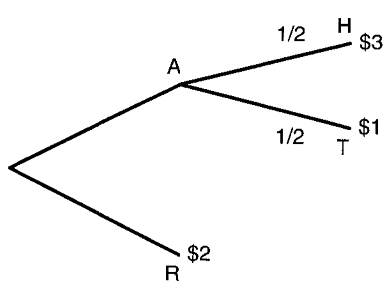

Figure 6.8 Risk aversion

steeper. Compute the expected utilities of accepting and rejecting the gamble. What should you do?

As all these examples suggest, the shape of your utility function relates to your attitude toward risk – or your **risk preference** – in the following way. Whether you should reject or accept a gamble with an expected value of zero depends on whether your utility function gets flatter or steeper (bends downwards or upwards). In general, we say that you are **risk averse** if you would reject a gamble in favor of a sure dollar amount equal to its expected value, **risk prone** if you would accept, and **risk neutral** if you are indifferent. Thus, you are risk averse if your utility function bends downwards (as you move from left to right), risk prone if your utility function bends upwards, and risk neutral if your utility function is a straight line.

Notice that the theory itself does not specify what the shape of your utility function should be. Most of the time, economists will assume that utility for money is increasing, so that more money is better. But that is an auxiliary assumption, which is no part of the theory. The theory does not constrain your attitude toward risk; it does not even say that your attitude toward risk has to be the same when you get more (or less money). For instance, you may have a utility function that looks like the solid line in Figure 6.9. Here, you are risk prone in the range below x^* and risk averse in the range above x^*. Or, you may have a utility function that looks like the dashed line. Here, you are risk averse in the range below x^* and risk prone in the range above x^*. The next exercise illustrates the manner in which attitudes toward risk are expressed in a variety of real-world behaviors.

Exercise 6.34 Attitudes to risk As far as you can tell, are the following people *risk prone, risk averse,* or *risk neutral*?

(a) People who invest in the stock market rather than in savings accounts.

(b) People who invest in bonds rather than in stocks.

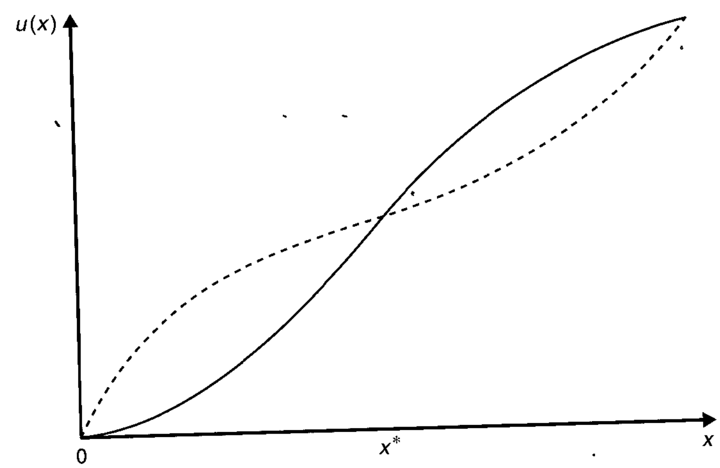

Figure 6.9 S-shaped utility functions

(c) People who buy lottery tickets rather than holding on to the cash.
(d) People who buy home insurance.
(e) People who play roulette.
(f) People who consistently maximize expected value.
(g) People who have unsafe sex.

Sometimes it is useful to compute the **certainty equivalent** of a gamble. The certainty equivalent of a gamble is the amount of money such that you are indifferent between playing the gamble and receiving the amount for sure.

Definition 6.35 Certainty equivalent *The certainty equivalent of a gamble G is the number CE that satisfies this equation:* $u(CE) = EU(G)$.

The certainty equivalent represents *what the gamble is worth to you*. The certainty equivalent determines your willingness-to-pay (WTP) and your willingness-to-accept (WTA). In graphical terms, suppose that you have to find the certainty equivalent given a utility function like that in Figure 6.10. Suppose the gamble gives you a 50 percent chance of winning A and a 50 percent chance of winning B.

1. Put a dot on the utility curve right above A. This dot represents the utility of A (on the y-axis).
2. Put another dot on the utility curve right above B. This dot represents the utility of B (on the y-axis).
3. Draw a straight line between the two dots.
4. Put an X half way down the straight line. The X represents the expected utility of the gamble (on the y-axis) and the expected value of the gamble (on the x-axis).
5. Move sideways from the X until you hit the utility curve.
6. Move straight down to the x-axis, and you have the certainty equivalent (on the x-axis).

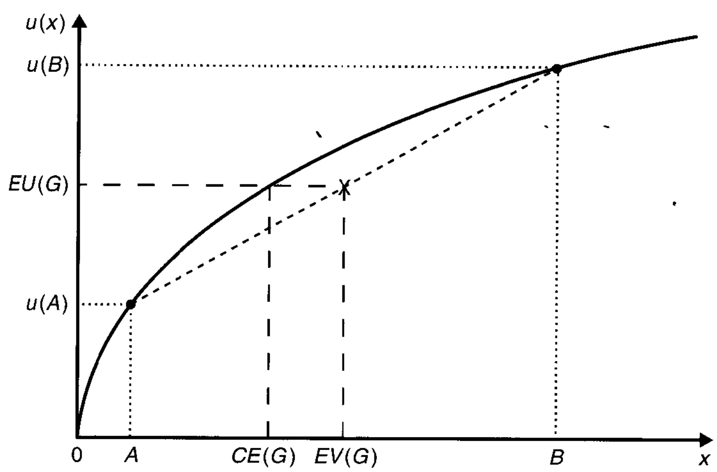

Figure 6.10 Finding the certainty equivalent

The procedure is illustrated in Figure 6.10. The same procedure can also be used for gambles where the probabilities are not 50–50. The only thing that changes is the placement of the X on the straight line in Figure 6.10. If there is, say, a 3/7 probability of winning A, and a 4/7 probability of winning B, starting from the left, you put that X four-sevenths of the way up from A to B. As the probability of winning B increases, the X will move right toward B and the expected utility of the gamble will approach the utility of B; as the probability of winning A increases, the X will move left toward A and the expected utility of the gamble will approach the utility of A. This makes sense.

Exercise 6.36 Certainty equivalents Demonstrate how to find the certainty equivalent of the same gamble in the case when the utility function bends upwards. Confirm that the certainty equivalent is greater than the expected value.

Remember: when the utility function bends downwards, you are risk averse, the dashed line falls below the utility function, and the certainty equivalent is less than the expected value. When the utility function bends upwards, you are risk prone, the dashed line falls above the utility function, and the certainty equivalent is greater than the expected value. Read this paragraph one more time to make sure you understand what is going on.

In algebraic terms, you get the certainty equivalent of a gamble by computing the expected utility x of the gamble, and then solving for $u(CE) = x$. Thus, you get the answer by computing $CE = u^{-1}(x)$. As long as $u(\cdot)$ is strictly increasing in money, which it ordinarily will be, the inverse function is well defined. Consider the gamble in Table 6.8. We know that the expected utility of this gamble is approximately 1.37. You get the certainty equivalent by solving for $u(CE) = 1.37$. Given our utility function $u(x) = \sqrt{x}$, this implies that $CE = 1.37^2 \approx 1.88$. Because you are risk averse, the certainty equivalent of the gamble is lower than the expected value of 2.

Example 6.37 St Petersburg paradox, cont. In Section 6.4 we learned that for an agent with utility function $u(x) = \log(x)$, the expected utility of the St Petersburg gamble is approximately 0.602. What is the certainty equivalent of the gamble?

We compute the certainty equivalent by solving the following equation: $\log(CE) = 0.602$. Thus, the certainty equivalent $CE = 10^{0.602} \approx 4.00$. That is, the St Petersburg gamble is worth \$4.

Exercise 6.38 Compute the certainty equivalent of the gamble in Figure 6.8, using the utility function $u(x) = x^2$.

We end this section with a series of exercises.

Exercise 6.39 Suppose that you are offered the choice between \$4 and the following gamble, G: 1/4 probability of winning \$9 and a 3/4 probability of winning \$1.

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(a) Suppose that your utility function is $u(x) = \sqrt{x}$. What is the utility of \$4? What is the utility of G? What is the certainty equivalent? Which would you choose?
(b) Suppose instead that your utility function is $u(x) = x^2$. What is the utility of \$4? What is the utility of G? What is the certainty equivalent? Which would you choose?

Exercise 6.40 Suppose that your utility function is $u(x) = \sqrt{x}$, and that you are offered a gamble which allows you to win \$4 if you are lucky and \$1 if you are not.
(a) Suppose that the probability of winning \$4 is 1/4 and the probability of winning \$1 is 3/4. What is the expected value of this gamble?
(b) Suppose that the probability of winning \$4 is still 1/4 and the probability of winning \$1 is 3/4. What is the expected utility of this gamble?
(c) Suppose that the probability of winning \$4 is still 1/4 and the probability of winning \$1 is 3/4. What is the certainty equivalent of the gamble; that is, what is the amount of money X such that you are indifferent between receiving \$$X$ for sure and playing the gamble?
(d) Imagine now that the probability of winning \$4 is p and the probability of winning \$1 is $(1 - p)$. If the utility of the gamble equals 3/2, what is p?

Exercise 6.41 Suppose that your utility function is $u(x) = \sqrt{x}$, and that you are offered a gamble which allows you to win \$16 if you are lucky and \$4 if you are not.
(a) Suppose that the probability of winning \$16 is 1/4 and the probability of winning \$4 is 3/4. What is the expected utility of this gamble?
(b) Suppose that the probability of winning \$16 is still 1/4 and the probability of winning \$4 is 3/4. What is the certainty equivalent of the gamble?
(c) Imagine now that the probability of winning \$16 is p and the probability of winning \$4 is $(1 - p)$. If the expected utility of the gamble equals 9/4, what is p?
(d) Are you risk averse or risk prone, given the utility function above?

Exercise 6.42 Lotto 6/49, cont. Compute the certainty equivalent of the Lotto 6/49 ticket from Exercise 4.28 if $u(x) = \sqrt{x}$.

6.6 Discussion

In this chapter, we have explored principles of rational choice under risk and uncertainty. As pointed out in the introduction to this chapter, according to the traditional perspective, you face a choice under uncertainty when probabilities are unknown or not even meaningful. In Section 6.2, we explored several principles of rational choice that may apply under such conditions. When it is both meaningful and possible to assign probabilities to the relevant states of the world, it becomes possible to compute expectations, which permits you to apply expected-value and expected-utility theory instead.

The distinction between risk and uncertainty is far from sharp. In real life, it may not be obvious whether to treat a decision as the one or the other.

Consider the regulation of new and unstudied chemical substances. Though there are necessarily little hard data on such substances, there is always some probability that they will turn out to be toxic. Some people argue that this means that policy-makers are facing a choice under uncertainty, that the maximin criterion applies, and that new chemicals should be banned or heavily regulated until their safety can be established. Others argue that we can and must assign probabilities to all outcomes, that the probability that new substances will turn out to be truly dangerous is low, and that expected-utility calculations will favor permitting their use (unless or until their toxicity has been established). Whether we treat a decision as a choice under uncertainty or under risk, therefore, can have real consequences. And it is not obvious how to settle such issues in a non-arbitrary way. (We will return to this topic under the heading of ambiguous probabilities in Section 7.5.)

One thing to note is that you cannot judge whether a decision was rational or not by examining the outcome alone. A rational decision, as you know, is a decision that maximizes expected utility given your beliefs at the time when you make the decision. Such a decision might lead to adverse outcomes. If something bad happens as a result of your decision, that does not mean you acted irrationally: you may just have been unlucky. Sometimes decision theorists use the term "right" to denote the decisions that lead to the best possible outcome. The fact that good decisions can have bad outcomes means that decisions can be rational but wrong. They can also be irrational but right, as when you do something completely reckless but see good results anyway; buying a lottery ticket as a means to get rich and winning truckloads of money might fall in this category. It goes without saying that we always want to make the right decision. The problem, of course, is that we do not know ahead of time which decision is the right one. That is why we aim for the *rational* decision – being the one with the greatest expectation of future utility.

Example 6.43 The rationality of having children It is sometimes argued that certain decisions cannot be made rationally. Philosopher L. A. Paul, for example, has argued that it is impossible to make a rational decision about having a child, because you cannot know ahead of time what it will be like, for you, to have a child. But such arguments may confuse the rational with the right. It is true that you cannot know what the right decision is: you may be very happy with a child, or you may be miserable. But you never know ahead of time what the right decision is. Luckily, ignorance is no obstacle to making a rational decision, as "rationality" is understood here. Rationality does not require that you know what anything is like – only that you choose whatever option maximizes expected utility given your beliefs at the time you are making the decision.

We will return to the topic of the right and the rational in Section 7.7.

Our study of the theory of choice under risk sheds further light on the economic approach to behavior as understood by Gary Becker, and in particular on what he had in mind when talking about maximizing behavior (see Section 2.8). Recall (from Section 4.7) that the standard approach does not assume that people consciously or not perform calculations in their heads: all talk about maximization (or optimization) is shorthand for the satisfaction of preferences. Notice also that this approach does not assume that people are omniscient, in the sense that they know what state of the world will obtain. What it does assume is that people assign probabilities to states of the world, that these probabilities satisfy the axioms of the probability calculus, that people assign utilities to outcomes, and that they choose that alternative which has the greatest expected utility given the probabilities and utilities.

In the next chapter, we consider some conditions under which these assumptions appear to fail.

ADDITIONAL EXERCISES

Exercise 6.44 *Deal or No Deal*, cont. You are on *Deal or No Deal* again, and you are facing three boxes. One of the three contains $1,000,000, one contains $1000, and one contains $10. Now the dealer offers you $250,000 if you give up your right to open the boxes.

(a) Assuming that you use expected value as your guide in life, would you choose the sure amount or the right to open the boxes?

(b) Assuming that your utility function $u(x) = \sqrt{x}$, and that you use expected utility as your guide in life, would you choose the sure amount or the right to open the boxes?

(c) Given the utility function, what is the lowest amount in exchange for which you would give up your right to open the boxes?

Exercise 6.45 The humiliation show You are on a game show where people embarrass themselves in the hope of winning a new car. You are given the choice between pressing a blue button and pressing a red button.

(a) If you press the blue button, any one of two things can happen: with a probability of 2/3, you win a live frog (utility −1), and with a probability of 1/3 you win a bicycle (utility 11). Compute the expected utility of pressing the blue button.

(b) If you press the red button, any one of three things can happen: with a probability of 1/9 you win the car (utility 283), with a probability of 3/9 you win a decorative painting of a ballerina crying in the sunset (utility 1), and with a probability of 5/9 you end up covered in green slime (utility −50). Compute the expected utility of pressing the red button.

(c) What should you do?

Exercise 6.46 Misguided criticism Some critics attribute to neoclassical economists the view that human beings have the ability to compute solutions to every maximization problem, no matter how complicated, in their heads and on the fly. For example: "Traditional models of unbounded rationality and optimization in cognitive science, economics, and animal behavior have tended to view decision-makers as possessing supernatural powers of reason, limitless knowledge, and endless time." Explain why this criticism misses the mark.

See also Exercise 7.34 on pages 176–177.

FURTHER READING

The classic definition and discussion of choice under uncertainty is Luce and Raiffa (1957, Ch. 13); Rawls (1971) defends Rawls's theory of justice and Harsanyi (1975) criticizes it. Helpful introductions to expected-utility theory include Allingham (2002) and Peterson (2009). The quip about not regretting one's mistakes appears in Wilde (1998 [1890], p. 34); *The Top Five Regrets...* is Ware (2012); and Kierkegaard (2000 [1843], p. 72) despaired of ever avoiding regret. The view that one cannot rationally decide whether to have children appears in Paul and Healy (2013). The critics are Todd and Gigerenzer (2000, p. 727).