Theories of Economic Growth - Solow Model

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Introduction

- Proposed by Robert Solow (1956) and Trevor Swan (1956)
- Most basic model to think about growth and macroeconomics, featuring:
  - dynamics: it can be discrete or continuous
  - general equilibrium
- Although not modelling
  - growth!
  - important macroeconomic correlates to growth like saving decisions
- Still, very useful to evaluate the role of
  - factor accumulation
  - technological progress

We'll cover the version in continuous time and with population growth.

Why continuous time?

Suppose

$$x(t+1) - x(t) = g(x(t))$$

- What is 1 period? One year? One week?
  - maybe we should make the time unit as small as possible!
- When t and $t+1$ are not too far apart, we can approximate the change as

$$\begin{aligned} x(t+\Delta t) - x(t) &\simeq \Delta t . g(x(t)) \\ \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) &\simeq g(x(t)) \end{aligned}$$

Setting

The model comprises:

- 1 closed economy
- 1 sector
  - representative firm producing output $Y(t)$ and selling it at $P(t) = 1$
  - *wheat economy*: Y can be consumed or used for more production
- consumers not explicitly modelled
  - no utility function
  - $C(t) = (1 - s)Y(t)$ with $s \in (0, 1)$
  - $L(t) = \bar{L}(t)$
  - $\bar{L}(t) = e^{n.t}.\bar{L}(0)$ with $\bar{L}(0) > 0$
    - $\Rightarrow \frac{\dot{L}(t)}{L(t)} = n$

Production

The production function F is **neoclassical**:

- twice differentiable and continuous
- features constant returns to scale wrt K and L:

$$F(A, \lambda K, \lambda L) = \lambda F(A, K, L)$$

- features diminishing marginal returns to each factor: $F_K > 0$, $F_{KK} < 0$, $F_L > 0$, $F_{LL} < 0$
- satisfies Inada conditions
  - $\lim_{K \to 0} F_K = \infty$, $\lim_{K \to \infty} F_K = 0$, and $F(0, L, A) = 0, \forall L$ and A
  - $\lim_{L \to 0} F_L = \infty$, $\lim_{L \to \infty} F_L = 0$, $\forall K$ and A

Production

Moreover:

- to hire production factors the firm pays rental rates $w(t)$ and $R(t)$.
- $K(0) \geq 0$
- capital depreciates at rate $0 < \delta \Rightarrow r(t) = R(t) - \delta$

Firm's optimization and Equilibrium

The representative firm maximizes

$$maxF(K(t), L(t), A(t)) - R(t)K(t) - w(t)L(t)$$

subject to $K(t) = \bar{K}(t) \geq 0$ and $L(t) = \bar{L}(t) \geq 0$.

Notice the problem is not dynamic!!

The FOC give

- $w(t) = F_L(K(t), L(t), A(t))$
- $R(t) = F_K(K(t), L(t), A(t))$

Equilibrium equations

The full list of equilibrium equations are:

- $\dot{K}(t) = sF(K(t), L(t), A(t)) - \delta K(t)$
- $Y(t) = C(t) + I(t)$
- $S(t) = I(t) = s.Y(t)$
- $w(t) = F_L(K(t), L(t), A(t))$
- $R(t) = F_K(K(t), L(t), A(t))$
- $r(t) = R(t) - \delta$
- $C(t) = (1 - s).Y(t)$
- $\bar{L}(t) = e^{n.t}$

Equilibrium definition

In the basic Solow model with

- population growth at rate n
- an initial capital stock $K(0)$
- and for a given sequence of $\{A(t)\}_{t=0}^{\infty}$

an **equilibrium path** is a sequence $\{K(t), L(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that:

- $K(t)$ satisfies $\dot{K}(t) = sF(K(t), L(t), A(t)) - \delta K(t)$
- $L(t) = e^{n.t}L(0)$
- $Y(t) = F(A(t), K(t), L(t))$
- $C(t) = (1-s).Y(t)$
- $w(t) = F_L(K(t), L(t), A(t))$
- $R(t) = F_K(K(t), L(t), A(t))$

Equilibrium

Assume no technological progress: $A(t) = A$.

Define $k(t) = K(t)/L(t)$, and
$y(t) = Y(t)/L(t) = F(k(t), 1, A) = f(k(t))$. Which implies that:

- $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - n$
- $\frac{\dot{y}(t)}{y(t)} = \frac{\dot{Y}(t)}{Y(t)} - n$

Then we obtain the *fundamental law of motion* of the Solow model:

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n + \delta)$$

The path for the rest of the variables follows from this law of motion.

Steady state definition

A steady-state equilibrium without technological progress is an equilibrium path in which $k(t) = k^* \forall t$.

Such equilibrium implies:

$$\begin{aligned} s.f(k^*) &= (\delta + n).k^* \\ \text{investment} &= \text{capital use (depreciation+ pop. growth)} \end{aligned}$$

Steady state existence and uniqueness

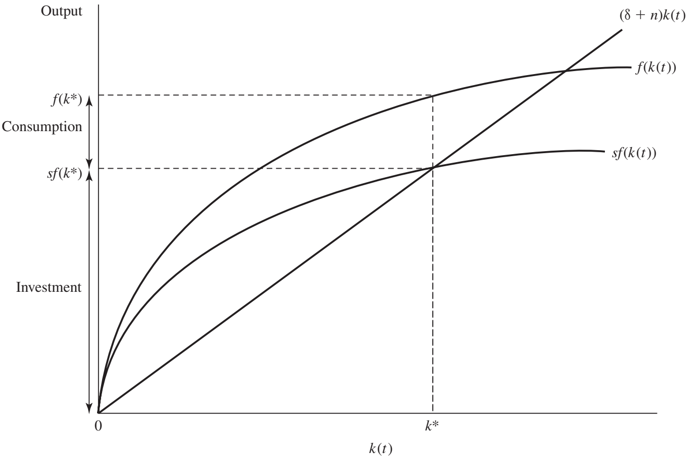

FIGURE 2.8 Investment and consumption in the steady-state equilibrium with population growth.

First results

We have a first idea of the potential determinants of differences in capital-labor ratios and output levels across countries:

- Different economies may have different SS's
- the level of y^{SS} is determined by parameters s, δ, and n.
- the same is true for A if assuming *Hicks-neutral* production function $\tilde{f}(k) = f(k)/A$
  - $\frac{\partial k^*(A,s,\delta,n)}{\partial A} > 0$, $\frac{\partial k^*(A,s,\delta,n)}{\partial s} > 0$, $\frac{\partial k^*(A,s,\delta,n)}{\partial \delta} < 0$, $\frac{\partial k^*(A,s,\delta,n)}{\partial n} < 0$
  - $\frac{\partial y^*(A,s,\delta,n)}{\partial A} > 0$, $\frac{\partial y^*(A,s,\delta,n)}{\partial s} > 0$, $\frac{\partial y^*(A,s,\delta,n)}{\partial \delta} < 0$, $\frac{\partial y^*(A,s,\delta,n)}{\partial n} < 0$

  Prove it!

  Are all of these results intuitive?
- c is similarly affected by A, n, and δ, but is not monotone in s

Maximum consumption level c^*_{gold}

Among the different values of s (which yield different SS's), there exists a s_{gold} such that c^* is at its highest level: c^*_{gold}. To find this notice that:

$$\begin{aligned} c^*(s) &= (1-s)f(k^*(s)) \\ sf(k^*(s)) &= (n+\delta)k^* \end{aligned}$$

So, $c^*(s) = f(k^*(s)) - (n+\delta)k^*(s)$. Then:

$$\begin{aligned} \frac{\partial c^*(s)}{\partial s} &= [f'(k^*(s)) - (n+\delta)]\frac{\partial k^*(s)}{\partial s} \\ \frac{\partial c^*(s)}{\partial s} &= 0 \Leftrightarrow f'(k^*(s_{gold})) = n+\delta \end{aligned}$$

Dynamic Inefficiency

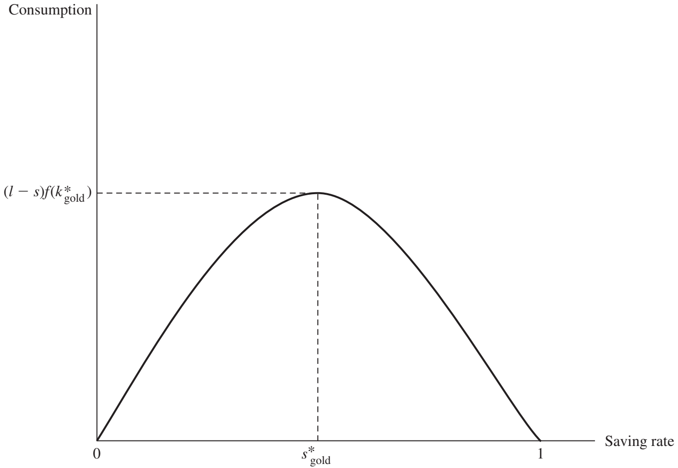

FIGURE 2.6 The golden rule level of saving rate, which maximizes steady-state consumption.

Stability-Theorem

Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Suppose that $g(x^*) = 0$ and that $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the non-linear differential equation $\dot{x}(t) = g(x(t))$, x^*, is **globally asymptotically stable**, that is, starting at any $x(0)$, $x(t) \to x^*$.

Transitional Dynamics in the Solow model

Under the current setting the model is globally asymptotically stable. Starting from any $k(0) > 0$, $k(t)$ monotonically converges towards k^*.

Why?

- F is continuously differentiable $\Rightarrow$ $\dot{k}$ is continuously differentiable
- if $k < k^* \Rightarrow s.f(k) - (n+\delta)k > 0$
- if $k > k^* \Rightarrow s.f(k) - (n+\delta)k < 0$

Transitional Dynamics in the Solow model

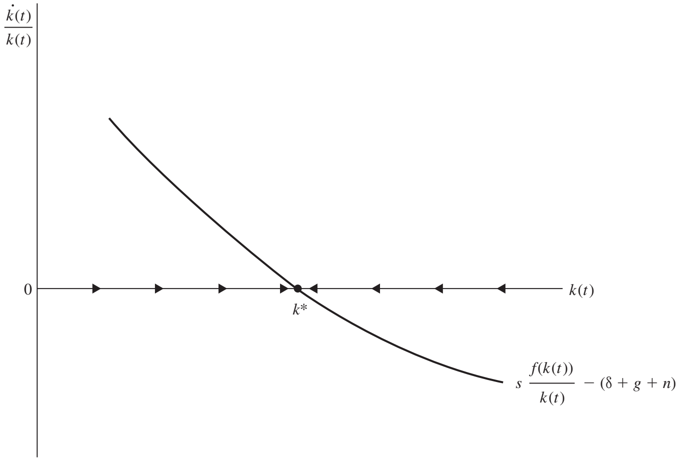

FIGURE 2.9 Dynamics of the capital-labor ratio in the basic Solow model.

Main conclusions of the basic Solow model

- the model has no growth! (only transitional growth until SS)
- to fix this we can
  - relax Assumptions 1 and 2 so we can have a model of sustained growth with no tech progress (AK)
  - recognize tech progress matters and introduce it to the basic model
    - How should we do it? Respecting the Kaldor facts!
      $\Rightarrow$ Balanced Growth

(Some of the) Kaldor facts

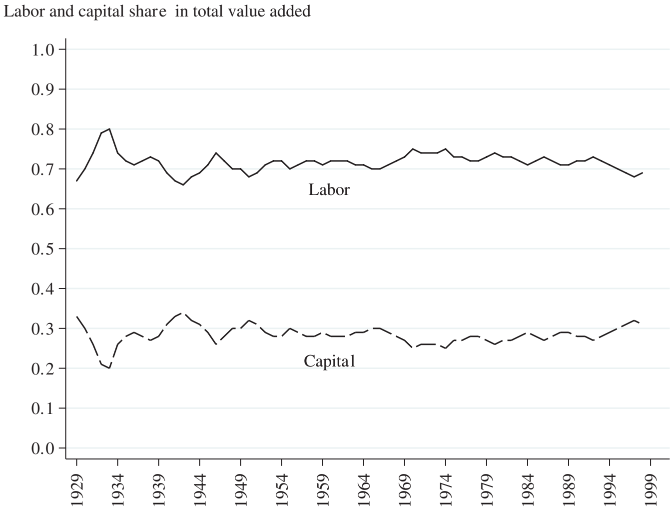

FIGURE 2.11 Capital and labor share in the U.S. GDP.

The Solow model with Technological Progress

The model is still simple, but more realistic since:

- A grows at rate g
- The economy grows! (and it's driven by A)
- Kaldor facts hold
    - growth is balanced: K/Y, R and factor shares in income are constant.

Types of Technological Progress

- Hicks-Nuetral: $A(t).F(K(t), L(t))$
- Solow-Neutral: $F(A(t).K(t), L(t))$ (Capital-augmenting)
- Harrod-Neutral: $F(K(t), A(t).L(t))$ (Labour-augmenting)

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Types of Technological Progress

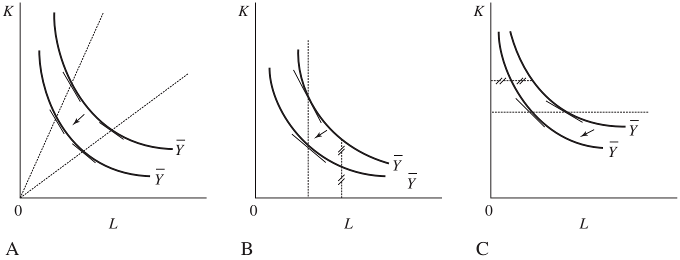

FIGURE 2.12 (A) Hicks-neutral, (B) Solow-neutral, and (C) Harrod-neutral shifts in isoquants.

Balanced Growth-Uzawa's Theorem

Consider a growth model with a general production function

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

where $\tilde{F}$ exhibits constant returns to scale in K and L. The aggregate resource constraint is

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t).$$

Suppose that there is constant population growth, $L(t) = e^{nt}L(0)$, and that there exists $T < \infty$ such that for all $t \geq T$, $\frac{\dot{Y}(t)}{Y(t)} = g_Y > 0$, $\frac{\dot{K}(t)}{K(t)} = g_K > 0$ and $\frac{\dot{C}(t)}{C(t)} = g_C > 0$. Then:

1. $g_Y = g_K = g_C$; and
2. for any $t \geq T$, there exists a function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ homogeneous of degree 1 in its two arguments, such that the aggregate production function can be represented as $Y(t) = F(K(t), L(t)A(t))$.

Uzawa's Theorem-Proof of Part 1

By assumption, for $t \geq T$ we have

- $Y(t) = e^{g_Y(t-T)}Y(T)$
- $K(t) = e^{g_K(t-T)}K(T)$
- $C(t) = e^{g_C(t-T)}C(T)$

so we can write the resource constraint as

$$\begin{aligned} (g_K + \delta)K(t) &= Y(t) - C(t) \\ (g_K + \delta)K(T) &= e^{(g_Y - g_K)(t-T)}Y(T) \\ &\quad - e^{(g_C - g_K)(t-T)}C(T) \end{aligned}$$

Differentiating wrt time:

$$\begin{aligned} 0 &= (g_Y - g_K)e^{(g_Y - g_K)(t-T)}Y(T) \\ &\quad -(g_C - g_K)e^{(g_C - g_K)(t-T)}C(T) \end{aligned}$$

Uzawa's Theorem-Proof of Part 1

$$
\begin{aligned}
0 &= (g_Y - g_K)e^{(g_Y - g_K)(t-T)}Y(T) \\
&\quad -(g_C - g_K)e^{(g_C - g_K)(t-T)}C(T)
\end{aligned}
$$

This expression holds if:

- $g_Y = g_K = g_C$
- $g_Y = g_C$ and $Y(T) = C(T)$, contradicts $g_K > 0$
- $g_Y = g_K$ and $C(T) = 0$, contradicts $g_C > 0$ being finite
- $g_K = g_C$ and $Y(T) = 0$, contradicts $Y(T) > 0$

Therefore it must be that $g_Y = g_K = g_C$.

Uzawa's Theorem-Proof of Part 2

For any $t > T$, the production function at T can be written as

$$e^{-g_Y(t-T)}Y(t) = \tilde{F}(e^{-g_K(t-T)}K(t), e^{-n(t-T)}L(t), \tilde{A}(T))$$
$$Y(t) = \tilde{F}(e^{(g_Y-g_K)(t-T)}K(t), e^{(g_Y-n)(t-T)}L(t), \tilde{A}(T))$$

From part 1, $g_Y = g_K$ for any $t \geq T$, so:

$$Y(t) = \tilde{F}(K(t), e^{(g_Y-n)(t-T)}L(t), \tilde{A}(T))$$

Since this equation is true for all $t > T$ and $\tilde{F}$ is H^1 in K and L, there exists a function F, that is H^1 such that

$$Y(t) = F(K(t), e^{(g_Y-n)t}L(t))$$

were the term in blue represents Harrod-Neutral technological change, so this can be re-written as

$$Y(t) = F(K(t), A(t)L(t)) \text{ with } \frac{\dot{A}(t)}{A(t)} = g_Y - n$$

Intuition of Uzawa's Theorem

Given the properties of the production function, if $g_K > 0$ then we must have $g_Y = g_K = g_C > 0$.

Then, if $n > 0$, **balanced growth** requires growth in A to compensate for $g_Y - n$.

- convenient to express these kind of models in *effective labor* units.

Notice the theorem is silent about factor prices!

- factor shares are not included in the discussion ($\rightarrow$ 2nd Uzawa T).

Comparative Dynamics

- Similar to Comparative Statics but analysing how the entire growth path reacts to changes in parameters
  - Consider a one-time, unanticipated and permanent increase in the saving rate from s to s'

Comparative Dynamics

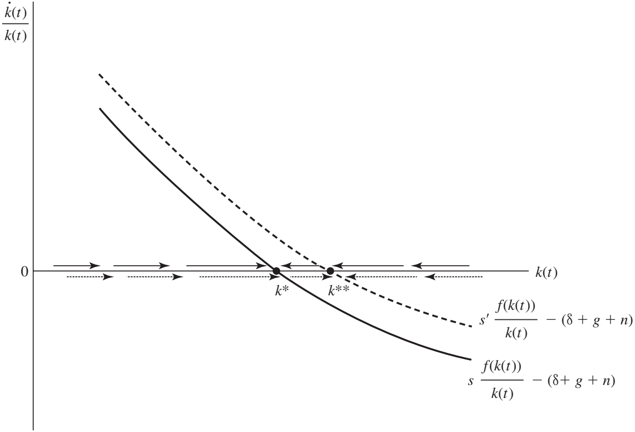

FIGURE 2.13 Dynamics following an increase in the saving rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.

The Solow Model and the data-Growth accounting

In Solow (1957), he asks the question:

- How much of growth can be attributed to increased L and K inputs?
  - Very little!
- The rest is explained by technological progress!

Growth accounting-Framework

$$
\begin{aligned}
Y &= F(A, K, L) \\
\frac{\dot{Y}}{Y} &= \frac{F_A A}{Y}\frac{\dot{A}}{A} + \frac{F_K K}{Y}\frac{\dot{K}}{K} + \frac{F_L L}{Y}\frac{\dot{L}}{L}
\end{aligned}
$$

Defining TFP as: $x = \frac{F_A A}{Y}\frac{\dot{A}}{A}$
Defining elasticities: $\epsilon_k = \frac{F_K K}{Y}$ and $\epsilon_l = \frac{F_L L}{Y}$
Assuming competitive markets: $w = F_L$ and $R = F_K$.
Then elasticities become factor shares: $\alpha_k = \epsilon_k$ and $\alpha_l = \epsilon_l$.

So we obtain:

$$
x = g_Y - \alpha_k g_K - \alpha_l g_L
$$

Growth accounting-Limitations

- the contribution of L is underestimated if we don't account for human capital
- the contribution of K is underestimated if prices used to aggregate them decline over time
- besides of many quality changes over time.

Then, the contribution of TFP will be overestimated!

The Solow Model and the data-Growth Regressions

Started by Barro (1991) and used extensively.

Solow model with labor-augmenting technological change:

$$\begin{aligned} y(t) &= A(t)f(k(t)) \\ \frac{\dot{y}(t)}{y(t)} &= g + \epsilon_k(k(t))\frac{\dot{k}(t)}{k(t)} \end{aligned}$$

and constant population growth

$$\frac{\dot{k}(t)}{k(t)} = \frac{s.f(k(t))}{k(t)} - (\delta + g + n) \tag{1}$$

The Solow Model and the data-Growth Regressions

$$\frac{\dot{k}(t)}{k(t)} = \frac{s.f(k(t))}{k(t)} - (\delta + g + n)$$

Log linearizing wrt $log(k)$ around steady state value k^*, which implies:

1. Deriving wrt $logk$ (using the property that $dg(x)/dlogx = (dg(x)/dx).x$): so we get $(f'.k - f).s/k$
2. Constructing the Taylor expansion of function $f(x)$ around $x = a$: $f(x) \approx f(a) + f'(a)(x - a)]$.
3. Using the fact that at SS: $sf/k = \delta + g + n$

$$\frac{\dot{k}(t)}{k(t)} \approx 0 + \left(\frac{f'(k^*)k^*}{f(k^*)} - 1\right)\frac{sf(k^*)}{k^*}(\log(k(t)) - \log k^*)$$

$$\frac{\dot{k}(t)}{k(t)} \approx (\epsilon_k(k^*) - 1)(\delta + g + n)(\log k(t) - \log k^*)$$

The Solow Model and the data-Growth Regressions

Merging both results together gives:

$$\frac{\dot{y}(t)}{y(t)} = g - (1 - \epsilon_k(k^*))(\delta + g + n)\epsilon_k(k^*)(\log k(t) - \log k^*)$$

Defining $y(t) = A(t)f(k(t))$ so $\log(y(t)) = \log A(t) + \log(f(k(t)))$ and so

$$\log(y(t)) \approx \log y^* + \frac{f'(k^*)}{f(k^*)} k^*(\log k(t) - \log k^*)$$

$$\log(y(t)) \approx \log y^* + \epsilon_k(k^*)(\log k(t) - \log k^*)$$

And merging this with the result above gives:

$$\frac{\dot{y}(t)}{y(t)} = g - (1 - \epsilon_k(k^*))(\delta + g + n)(\log y(t) - \log y^*)$$

The Solow Model and the data-Growth Regressions

$$\frac{\dot{y}(t)}{y(t)} = g + (1 - \epsilon_k(k^*))(\delta + g + n)(\log y^* - \log y(t))$$

This equation shows:

- 2 sources of growth in per capita income in the Solow model:
  - technological progress: g
  - convergence: $y(t) < y^*$
- speed of convergence depends on:
  - rate at which the effective capital-labor ratio needs to be replenished: $\delta + g + n$
  - capital-elasticity of the production function: $\epsilon_k(k^*)$

The Solow Model and the data-Growth Regressions

An approximation with observables:

$$g_Y = \frac{\Delta y_i(t' - t)}{y_i(t)} = b^0 + b^1 . \log y_i(t) + \nu_i$$

The evidence shows:

- speed of convergence coefficient b^1 is significantly negative for developed economies
- the same is not true for a sample of entire world.

Still, unconditional convergence is not what Solow predicts!

The Solow Model and the data-Growth Regressions

A more sensible approach is estimating:

$$\frac{\Delta y_i(t'-t)}{y_i(t)} = b_i^0 + b^1 . \log y_i(t) + \nu_i$$

i.e. allowing for country specific tech progress g. or even

$$\frac{\Delta y_i(t'-t)}{y_i(t)} = b_i^0 + b_i^1 . \log y_i(t) + \nu_i$$

i.e. allowing also for country specific SS determinants (not used in the literature).

While informative, regressions like these have problems (most notably, endogeneity).