Theories of Economic Growth - AK models and convergence

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Intro

- So far we've covered models were growth is the outcome of exogenous tech progress.
- These are useful to explain income differences for economies with access to the same technologies.
- Not useful to explain different technologies.
- Not useful to explain long-term growth.
- We need endogenous technology choices and technological progress!
- Today we show that sustained growth can be achieved in a (quasi-)neoclassical framework.

The AK-model

- Sets the following production function:

$$Y(t) = AK(t)$$

- By eliminating diminishing returns to K, sustained growth can be achieved by accumulating K.

The Setup-Consumers

- CRRA preferences: $u(c(t)) = \frac{c(t)^{1-\sigma}-1}{1-\sigma}$
- exogenous population growth rate: n
- inelastic labour supply: $L(t) = \bar{L}(t)$
- exogenous discount factor: ρ

Consumers' problem:

$$\max_{c(t),a(t)} \quad \int_0^\infty e^{-(\rho-n)t} u(c(t))dt$$

$$s.t. \quad \dot{a}(t) = [r(t) - n]a(t) + w(t) - c(t)$$

$$\lim_{t\to\infty} a(t)e^{-\int_0^t [r(s)-n]ds} = 0$$

FOC: $\Rightarrow$ Euler equation: $\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}[r(t) - \rho]$

The Setup-Producers

- Production function $Y(t) = AK(t)$
  - In per capita terms $y(t) = Ak(t)$
  - Inada conditions no longer hold
  - $f'(k) = A \forall t$, so $\lim_{k \to \infty} f' = A > 0$.

- $r(t) = R(t) - \delta$
- FOC gives: $R(t) = R = A$

  $\Rightarrow r(t) = r = A - \delta$

Equilibruim

- $k(t) = a(t)$
- We can re-write:
  - LMK: $\dot{k}(t) = [A - \delta - n]k(t) - c(t)$
  - EEq: $\frac{\dot{c}}{c} = \frac{1}{\sigma}[A - \delta - \rho] \rightarrow$ Constant!
  - TVC: $\lim_{t\rightarrow\infty} k(t)e^{-[A-\delta-n]t} = 0$
- We assume $A > \delta + \rho$ to have positive growth.
- The growth rate of consumption doesn't depend on $k(t)$
  - Starting at $k(0)$, c will immediately grow at its constant rate.

- We can show that something similar happens to $k(t)$ and $y(t)$.

  $\Rightarrow$ No transitional dynamics in this model.

No Transitional Dynamics

Let us know show that $k(t)$ grows at a constant rate:

- Call: $g = A - \delta - n$ and $g^* = (A - \delta - \rho)/\sigma$. Assume $g > g^*$.
- Write $c(t) = c(0)e^{g^* t}$.
- Rewrite the LMK as $\dot{k}(t) - g.k(t) = c(0)e^{g^* t}$
- Solve this first-order non-autonomous linear differential equation.
- We obtain $k(t) = \kappa.e^{g.t} + \frac{c(0)e^{g^* t}}{g - g^*}$, where κ is a constant tbd.
- From this, it looks like the growth rate of $k(t)$ is not constant. But:

No Transitional Dynamics

- Look at the TVC:

$$\lim_{t\to\infty}\left[\kappa.e^{g.t}+\frac{c(0)e^{g^*t}}{g-g^*}\right]e^{-gt}=\lim_{t\to\infty}\left[\kappa+\underbrace{\frac{c(0)e^{-(g-g^*).t}}{g-g^*}}_{\to 0}\right]=0$$

- TVC can only hold if $\kappa = 0$.
  - which implies that $k(t)=\frac{c(0)e^{g^*t}}{g-g^*}$
    $\Rightarrow$ growth rate of $k(t)$ is $g^*=(A-\delta-\rho)/\sigma$
    $\Rightarrow c(0)=k(0).(g-g^*)$

- With AK technology, the growth rates of $k(t)$ and $y(t)$ are equal.

Results

- Sustained growth is endogenous
  - in the sense that it results from the model's parameters.
- The saving rate is endogenous.

$$s = \frac{\dot{K}(t) + \delta K(t)}{Y(t)} = \frac{\dot{k}(t)/k(t) + n + \delta}{A} = \frac{A - \rho + \sigma n + (\sigma - 1)\delta}{A\sigma}$$

- The competitive equilibrium is Pareto optimal.

Shortcomings of the AK-model

- Knife edge case.
- K is the only important factor of production. It's share in national income converges to 1.
  - In the data we've seen its closer to $1/3$!
- there is no technological progress.
  - We don't see growth being led by k accumulation in the data.
- No transitional dynamics $\Rightarrow$ no convergence.
- Small differences in country's parameter generate ever increasing differences in income.

Learning by doing interpretation of the AK-model

- While the AK model might be hard to interpret, we can show that a simple model with learning-by-doing approximates an AK specification.
- Take Romer (1986), widely considered the first endogenous growth model.

The Setup

- Knowledge accumulation is a by-product of aggregate capital accumulation: *technological spillovers*
- The economy consists of a set $[0, 1]$ of firms denoted by i, with:

$$
\begin{aligned}
Y_i(t) &= F(K_i(t), A(t)L_i(t)) \\
\int_0^1 K_i(t)di &= K(t) \\
\int_0^1 L_i(t)di &= L
\end{aligned}
$$

- F satisfies Inada conditions
- Learning-by-doing: $A(t) = BK(t)$

  $\Rightarrow$ Aggregate production function is: $Y(t) = F(K(t), BK(t)L)$

The Setup

$$Y(t) = F(K(t), BK(t)L)$$

- H^1: $\frac{Y(t)}{K(t)} = F(1, BL) = \tilde{f}(L)$
- per capita: $y(t) = \frac{Y(t)}{L} = \frac{Y(t)}{K(t)}\frac{K(t)}{L} = \tilde{f}(L)k(t)$

- competitive markets:

$$\begin{aligned} w(t) &= \frac{\partial F}{\partial L} = K(t)\tilde{f}'(L) \\ R(t) &= \frac{\partial F}{\partial K}\Big|_{dA=0} = \tilde{f}(L) - L\tilde{f}'(L) = R \\ r &= R - \delta \end{aligned}$$

Competitive Equilibrium

Definition

- Euler equation: $g_C^* = \frac{1}{\sigma}[r - \rho]$
- LMK: $\dot{k}(t) = [\tilde{f}(L) - \delta]k(t) - c(t)$
- TVC: $\lim_{t \to \infty} k(t)e^{-[\tilde{f}(L)-\delta]t} = 0$

- $g_C^* = g_K^* = g_Y^* = g_A^*$
- Assume $r > \rho$ for positive growth.

Results

- Technology A endogenously grows at a constant rate as the result of firm's investing decisions.

Shortcomings of the Romer86 model

- No diminishing returns to capital accumulation.
- Small differences in country's parameter generate large and increasing differences in income.
  - quantitatively differences in growth rates might be too large.
  - at odds with a relatively stable distribution of income (post-war).
- Scale effect.
  - Population must be constant, otherwise explosive growth
- In the competitive equilibrium, investment is sub-optimal:
  - Private returns to investing: $\frac{\partial Y}{\partial K}|_{dA=0} = R = \tilde{f}(L) - L\tilde{f}'(L)$
  - Social returns to investing: $\frac{\partial Y}{\partial K} = \tilde{f}(L)$

Convergence

- Important quantitative differences in terms of convergence between neoclassical and AK growth models.
  - Convergence is too fast in the neoclassical setting and too slow (non-existent) in the AK model.

Solow model

$$\frac{\dot{y}(t)}{y(t)} = g + \underbrace{(1 - \epsilon_k(k^*))(\delta + g + n)}_{\lambda}(\log y^* - \log y(t))$$

Cobb Douglas Technology: $Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha} \Rightarrow \epsilon_k = \alpha$

$$\lambda = (1 - \alpha)(\delta + g + n)$$

Plausible parameter values:

- $\alpha = 1/3$
- $g = 0.02$
- $n = 0.01$
- $\delta = 0.05$

$\Rightarrow$ convergence speed $\lambda = 0.054$

Solow model-drawbacks

- Empirical studies show that the speed of convergence should be around 0.02
- Convergence predicted by original Solow model is too fast!
- Can be toned-down by including human capital (see Mankiw, Romer and Weil, 1992)
    - But results from this approach are at odds with micro evidence on returns to schooling.

Introducing convergence into the AK model

Two versions of the AK model where we have convergence.

- Distance to frontier

  Main idea: bring AK closer to neoclassical growth by reducing returns to capital accumulation

- Armington

  Main idea: show that even with constant returns to physical capital accumulation, countries can experience diminishing value of capital.

Distance to the frontier model

- Two countries: Advanced (A) and Backward (B)
- Capital in A grows at rate g^*
- Production function in B: $Y_B = A_B K_B^{\alpha} L_B^{1-\alpha} \underbrace{\rightarrow}_{L_B=1} Y_B = A_B K_B^{\alpha}$
- LbD+IntSpillovers: $A_B = a_B K_B^{1-\alpha-\psi} K_A^{\psi}$
- Fixed investment rate: $\dot{K}_B = s_B Y_B - \delta K_B$
- Define $K_R = K_A / K_B$

$$\Rightarrow \frac{\dot{K}_B(t)}{K_B(t)} = s_B a_B K_R(t)^{\psi} - \delta$$

$$\Rightarrow \frac{\dot{K}_R(t)}{K_R(t)} = g^* - \frac{\dot{K}_B(t)}{K_B(t)} = g^* - s_B a_B K_R(t)^{\psi} + \delta \equiv G(K_R(t))$$

Distance to the frontier model

$$\frac{\dot{K}_R(t)}{K_R(t)} = g^* - s_B a_B K_R(t)^{\psi} + \delta \equiv G(K_R(t))$$

- Stable and unique solution: $K_R^* = \left[\frac{g^*+\delta}{s_B a_B}\right]^{\frac{1}{\psi}} \Rightarrow \lim_{t\to\infty} \frac{\dot{K}_B(t)}{K_B(t)} = g^*$
- Adjustment of speed (near the SS): $\lambda = -G'(K_R^*).K_R^*$
  $\Rightarrow \lambda = \psi(g^* + \delta)$
- with $g^* = 0.02$ and $\delta = 0.05$, we need $\psi = 1/4$ to get $\lambda = 0.02$

Armington

- Two countries: home () and foreign (f).
- Each country produces output (welfare) following:

$$Y = K^{\alpha} X^{\frac{1-\alpha}{2}} X_f^{\frac{1-\alpha}{2}}$$

- X is an intermediate input produced using one unit of output.
- Trade happens in intermediate goods X and X_F.
- $P_Y = 1$ & perfectly competitive market of $X \Rightarrow P_X = 1$
- $P_{Y_f} = P_{X_f} = P_f$ is given.
- terms of trade for home are defined as the ratio $1/P_f$

Armington

- Domestic producers maximize:

$$\max_{X,X_f}[K^{\alpha}X^{\frac{1-\alpha}{2}}X_f^{\frac{1-\alpha}{2}} - X - p_f X_f]$$

$\Rightarrow X = \left[\frac{1-\alpha}{2}\right] Y \ , \ P_f X_f = \left[\frac{1-\alpha}{2}\right] Y$

- Taking this back into the production function:

$$Y = \underbrace{K \left[\frac{1-\alpha}{2}\right]^{\frac{1-\alpha}{\alpha}} [P_f]^{-\frac{1-\alpha}{2\alpha}}}_{A}$$

- $g = \frac{\dot{K}}{K} = sA - \delta$

Armington

$$g = \frac{\dot{K}}{K} = s\left[\frac{1-\alpha}{2}\right]^{\frac{1-\alpha}{\alpha}} [P_f]^{-\frac{1-\alpha}{2\alpha}} - \delta$$

- the growth rate depends positively on terms of trade
  - If initially the domestic growth rate exceeds the world growth rate ($g > \bar{g}$), the foreign demand for X will not grow as fast as the country's demand for X_f
  - P_f must increase to preserve trade balance.
  - This will tend to bring the domestic country's growth rate down to the world level.
- there is convergence!