Teorías del Crecimiento Económico
Lecture Notes on Unified Growth Theory

Guzmán Ourens
Tilburg University

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Main Reference

Galor, O. 2005 "From Stagnation to Growth: Unified Growth Theory", in Aghion, P. and Durlauf, S.N. (eds.), Handbook of Economic Growth, North Holland, Elsevier Academic Press, Vol. 1A, pp. 171-293.

1 Introduction

Living standards enjoyed by people in the developed world today were not the result of constantly increasing possibilities over time. Instead, economic growth was relatively non-existent for the most part of human history and only took off in the last couple of centuries.

The transition from stagnation to growth was at the same time explosive (looking at it from a historical perspective) but also relatively gradual (for people living in that era). The first statement implies that the pattern is hard to reconcile with endogenous growth models, while the latter makes exogenous growth models unsuitable for the task. Therefore, a Unified Growth Theory is needed. Oded Galor and co-authors proposed such theory in the hopes that understanding how currently rich economies started growing could shed some light on what are the challenges for less developed economies.

The theory presented by Galor will reconcile the following facts:

- For almost all human history we had Malthusian stagnation:
  - technological progress and population growth was low.
  - the average rate of income per capita growth was close to zero.
  - human capital formation was close to zero.
  - stable birth and mortality rates (absent exogenous shocks).
- In the past two centuries, many regions escaped stagnation:
  - technological progress and population growth increased drastically.

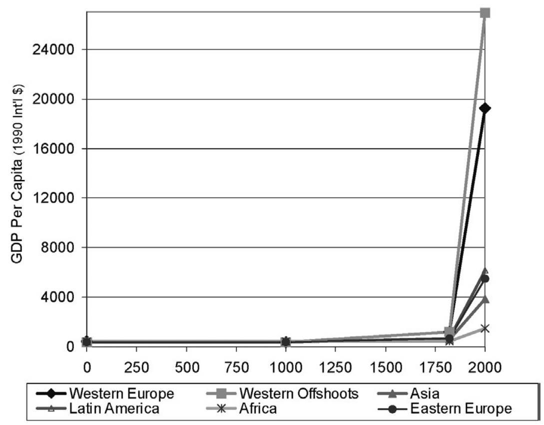

Figure 1. The evolution of regional income per capita over the years 1–2000. Source: Maddison (2003).[2]

- income per capita growth took off.
- human capital formation took off.
- demographic transition.

The establishment of a unified growth theory required major methodological innovations in the construction of dynamical systems. A theory in which economies take off gradually but swiftly from an epoch of a stable Malthusian stagnation would necessitate a gradual escape from an absorbing (stable) equilibrium – a contradiction to the essence of a stable equilibrium. One of the key methodological contributions of the UGT is to show that the observed rapid, continuous, phase transition would be captured by a single dynamical system, if the set of steady-state equilibria and their stability would be altered qualitatively in the process of development.

2 Facts

In this section, history will be split into three parts: the Malthusian Epoch, the Post-Malthusian Regime and the Sustained Growth Regime. The exact duration of each depends on the region of the world, but taking the developed world perspective we can think of the first period lasting until the end of the 18th century, the second lasting until the end of the 19th century, and the third still ongoing.

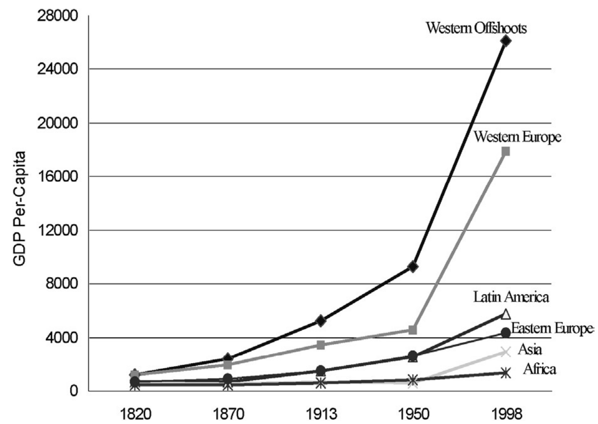

Figure 32. The great divergence. Source: Maddison (2001).

2.1 The Malthusian Epoch

The Malthusian epoch is characterized by:

- low technological progress and population growth.
- the little technological progress that increased income levels translated into higher population so income per capita was constant.
- human capital formation was close to zero.
- no demographic transition

People lived at subsistence levels during most of human history. The main reason for this was that higher income in one region translated into more people living there, keeping living conditions pretty much constant. That does not mean that technological progress was non-existent. Some existed and translated into denser population in certain regions of the world, notably Western Europe and East Asia. The average growth rate of output per capita over this period ranged from 0% in the impoverished region of Africa to a sluggish rate of 0.14% in the prosperous region of Western Europe.

Moreover, cycles were frequent and drastic as can be observed in the case of England. Per capita GDP declined during the 13th century, and increased sharply during the 14th and 15th centuries in response to the catastrophic population decline in the aftermath of the Black Death.

Fluctuations in population and wages over this epoch exhibited a clear Malthusian pattern. Episodes of technological progress, land expansion, favourable climatic conditions, or major epidemics (that resulted in a decline of the adult population), brought about a temporary increase in real wages and income per capita. Ultimately, however, most of this increase in real resources per capita was channelled towards increasing population and reducing per capita income.

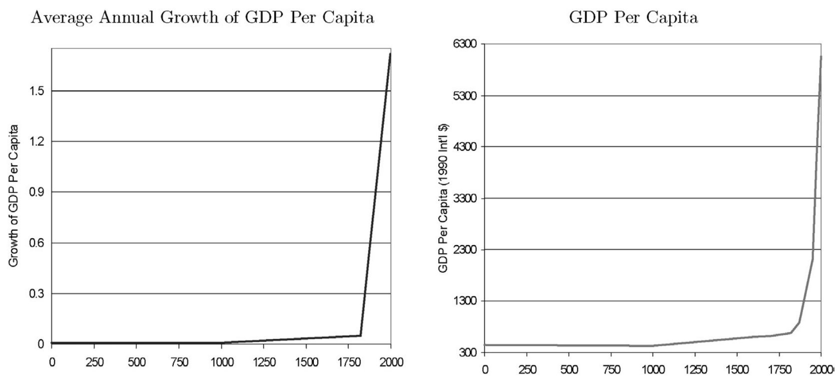

Figure 2. The evolution of the world income per capita over the years 1–2000. Sources: Maddison (2001, 2003).

2.2 The Post-Malthusian Regime

This period is characterized by:

- technological progress markedly increased along with industrialization.
- rising income per capita.
- high population growth due to increasing income levels.
- human capital starting to increase.
- the demographic transition starts

Technological progress and industrialization spurred the Industrial Revolution at the beginning of the 19th century. Over the course of the century that followed the process expanded out of Western Europe reaching mostly the Western Offshoots (Australia, Canada, New Zealand, and the US).

The Malthusian mechanism was still in place so higher income implies higher population growth.

The difference with respect to the previous period is that technological progress is so strong that population cannot catch up so income per capita increases. The average growth rate of output per capita in the world soared from 0.05% per year in the time period 1500-1820 to 0.53% per year in 1820-1870, and 1.3% per year in 1870-1913.

The timing of the take-off and its magnitude differed across regions. This created very large differences in living conditions in different regions of the

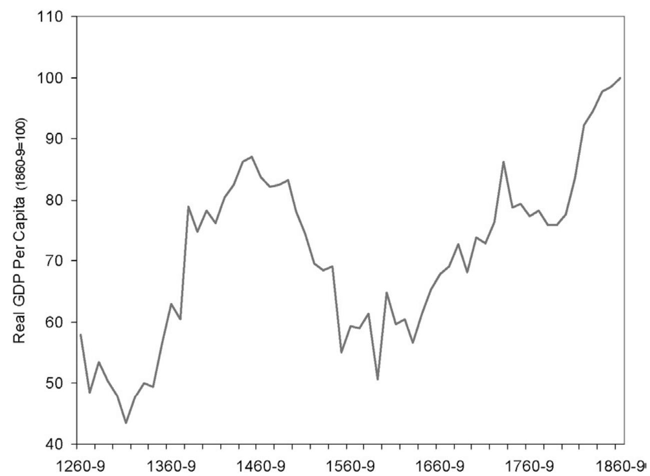

Figure 3. Fluctuations in real GDP per capita: England, 1260–1870. Source: Clark (2001).

world, for the first time in history. Technological leaders and land-abundant regions during the Post-Malthusian era improved their relative position in the world in terms of their level of income per capita as well as their population size. The increase in population density of technological leaders persisted as long as the positive relationship between income per capita and population growth was maintained. The acceleration in technological progress and the accumulation of physical capital and to a lesser extent human capital, generated a gradual rise in real wages in the urban sector and (partly due to labour mobility) in the rural sector as well.

The rate of population growth relative to the growth rate of aggregate income declined gradually over the period. For instance, the growth rate of total output in Western Europe was 0.3% per year between 1500 and 1700, and 0.6% per year between 1700 and 1820. In both periods, two thirds of the increase in total output was matched by increased population growth, and the growth of income per capita was only 0.1% per year in the earlier period and 0.2% in the later one.

The acceleration in technological progress during the Post-Malthusian Regime and the associated increase in income per capita stimulated the accumulation of human capital in the form of literacy rates, schooling, and health. The increase in the investment in human capital was induced by the gradual relaxation in households' budget constraints (as reflected by the rise in real wages and income per capita), as well as by qualitative changes in the economic environment that increased the demand for human capital and induced households to invest in

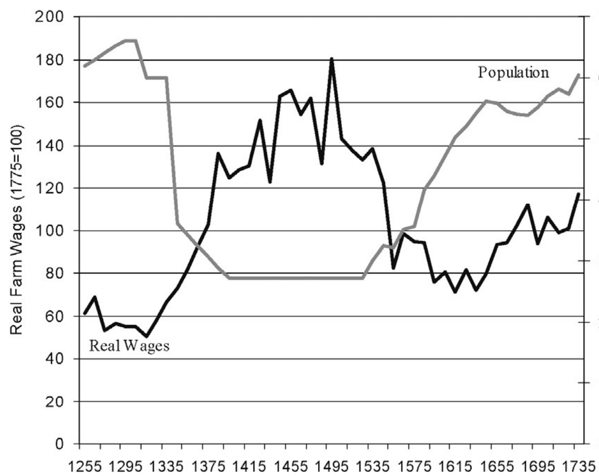

Figure 6. Population and real wages: England, 1250–1750. Sources: Clark (2001, 2002).

the education of their offspring.

In the first phase of the Industrial Revolution, human capital had a limited role in the production process. Education was motivated by a variety of reasons, such as religion, enlightenment, social control, moral conformity, socio-political stability, social and national cohesion, and military efficiency. The extensiveness of public education was therefore not necessarily caused by industrial development and it differed across countries due to political, cultural, social, historical and institutional factors. In the second phase of the Industrial Revolution, however, the demand for education increased, reflecting the increasing skill requirements in the process of industrialization.

2.3 The Sustained growth Regime

This period is characterized by:

- technological progress continues.
- demographic transition is completed.
- explosive and sustained growth in per capita income.
- high human capital formation.

The transition of the developed regions of Western Europe and the Western Offshoots to the state of sustained economic growth occurred towards the end of the 19th century, whereas the transition of some less developed countries in

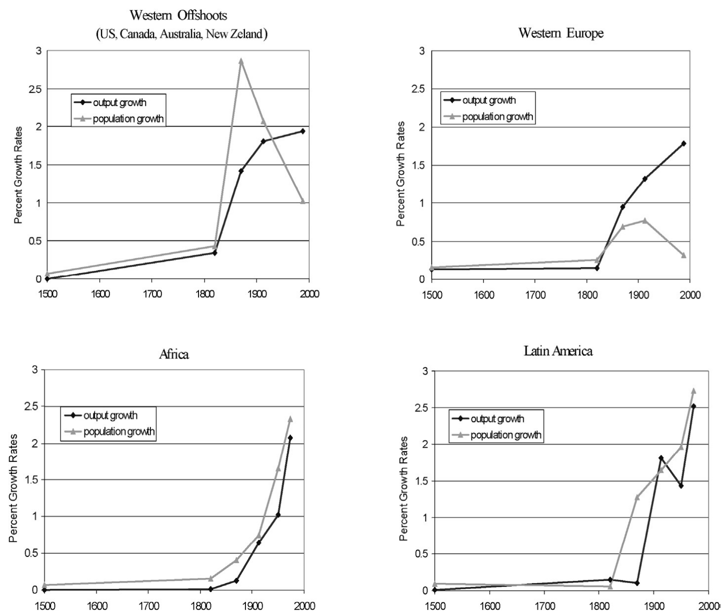

Figure 11. Regional growth of GDP per capita and population: 1500–2000. Source: Maddison (2001).

Asia and Latin America occurred towards the end of the 20th century. Africa, in contrast, is still struggling to make this transition.

The acceleration in technological progress and the associated rise in the demand for human capital brought about a demographic transition (quality-quantity decision on offspring tilts towards quality) in Western Europe, Western Offshoots, and in many of the less advanced economies, permitting sustained increase in income per capita.[1] Income per capita in the last century has advanced at a stable rate of about 2% per year in Western Europe and the Western Offshoots.[2]

Over the time period 1890–1999, the contribution of human capital accumulation to the growth process in the US nearly doubled whereas the contribution of physical capital declined significantly.

The unprecedented increase in population growth during the Post-Malthusian Regime was ultimately reversed and the demographic transition brought about a significant reduction in fertility rates and population growth in various regions of the world, enabling economies to convert a larger share of the fruits of factor

[1] Both in developed and less developed regions, the onset of the process of human capital accumulation preceded the onset of the demographic transition, suggesting that the rise in the demand for human capital in the process of industrialization and the subsequent accumulation of human capital played a significant role in the demographic transition and the transition to a state of sustained economic growth.

[2] Such high and sustained growth for economies in this regime, enlarged the gap between these economies and those that did not move towards this regime yet. Notice that the ratio of GDP per capita between the richest region and the poorest region in the world was only 1.1:1 in the year 1000, 3:1 in 1820 and 18:1 ratio in 2001.

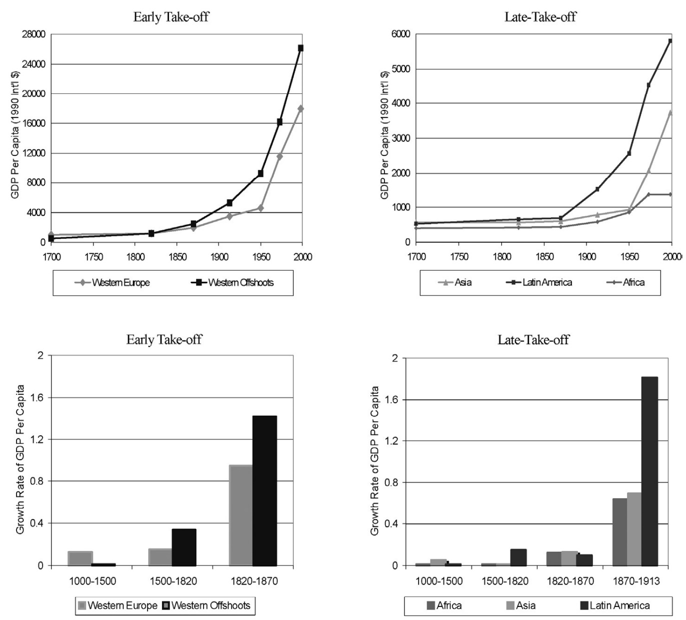

Figure 9. The differential timing of the take-off across regions. Source: Maddison (2001).

accumulation and technological progress into growth of income per capita. The demographic transition enhanced the growth process via three channels: (a) the reduction of the dilution of the stock of capital and land; (b) the enhancement of investment in human capital; (c) the alteration of the age distribution of the population, temporarily increasing the size of the labour force relative to the population as a whole.

The timing of completion of the demographic transition varies greatly: it was at the end of the 19th century in Europe and the Offshoots, one century later for Latin America and Asia (1970's), and it hasn't been completed in Africa yet.

Life expectancy rises as the result of declining mortality rates.

In the first phase of the Industrial Revolution, prior to the implementation of significant education reforms, physical capital accumulation was the prime engine of economic growth. In the absence of significant human capital formation, the concentration of capital among the capitalists widened wealth inequality. Once education reforms were implemented, however, the significant increase in the return to labour relative to capital, as well as the significant increase in the real return to labour and the associated accumulation of assets by the workers, brought about a decline in inequality.

3 The UGT framework

The UGT is built around the following pillars

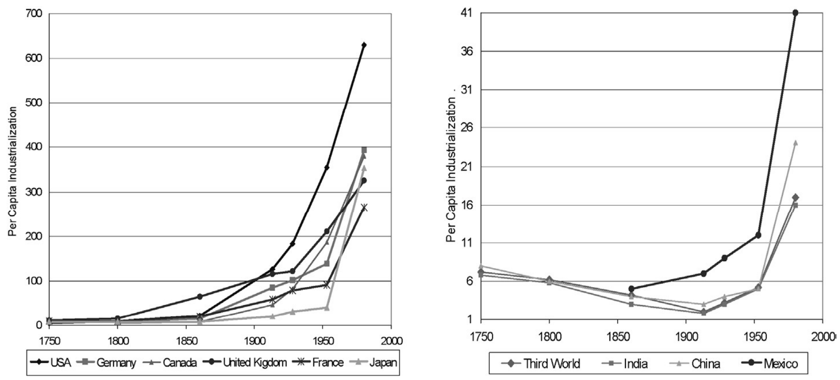

Figure 14. Per capita levels of industrialization: (UK in 1900 = 100). Source: Bairoch (1982).[24]

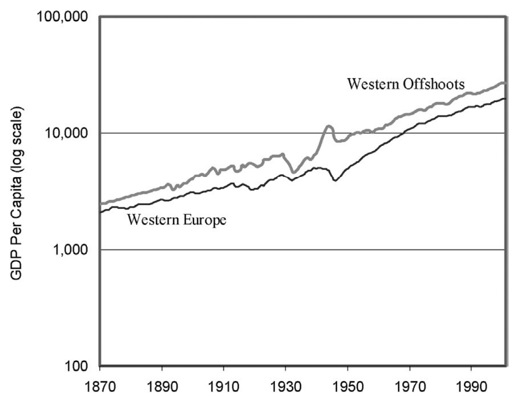

Figure 17. Sustained economic growth: Western Europe and the Western Offshoots, 1870–2001. Source: Maddison (2003).

- Malthusian relationship between income and population.

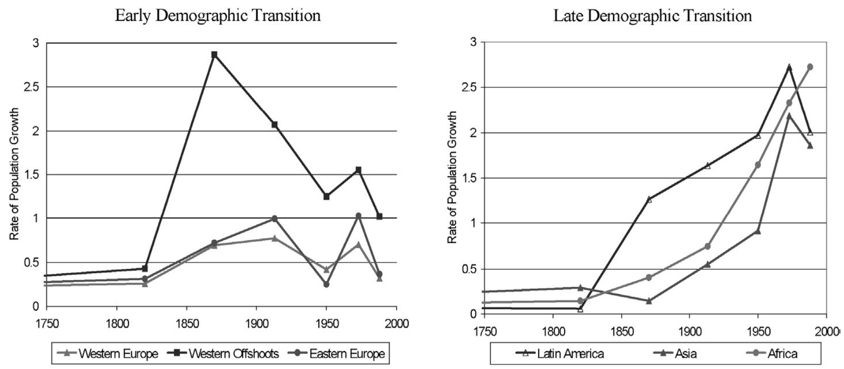

Figure 20. The differential timing of the demographic transition across regions. Source: Maddison (2001).

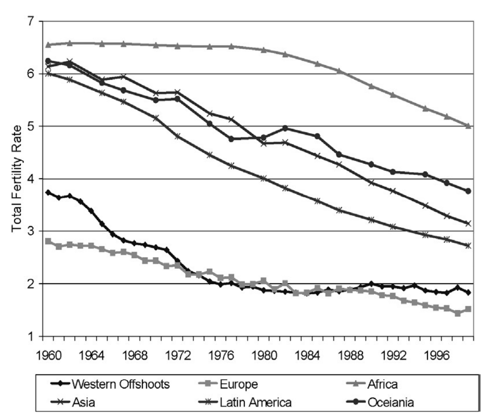

Figure 21. The evolution of Total Fertility Rate across regions, 1960–1999. Source: World Development Indicators (2001).

- Positive effect from population to innovation.[3]

[3]In the Malthusian era, the technological frontier was not distant from the working envi-

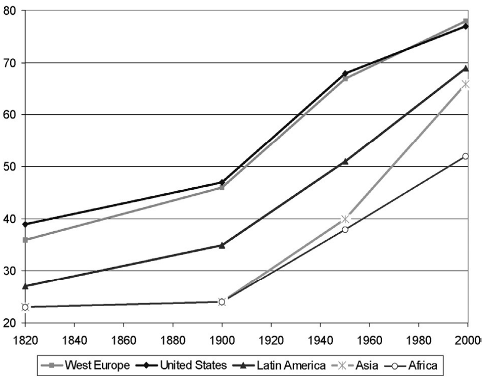

Figure 27. The evolution of life expectancy across regions, 1820–1999. Source: Maddison (2001).

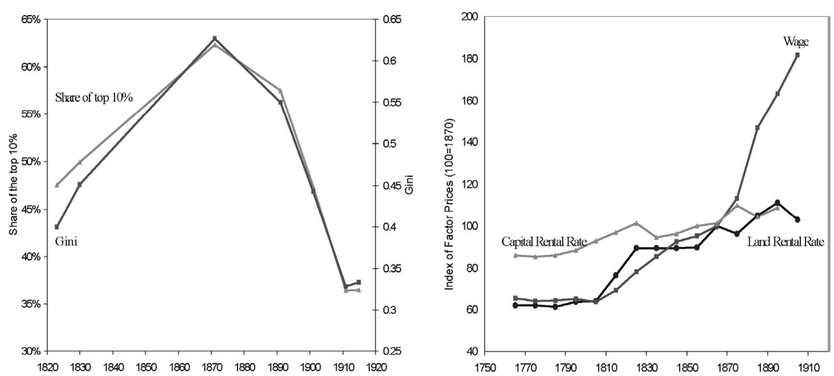

Figure 29. Wealth inequality and factor prices: England 1820–1920. Sources: Williamson (1985) for inequality and Clark (2002, 2003) for factor prices.

- Positive effect from innovation to human capital formation.
- Endogenous population growth: households decide rationally how many

ronment of most individuals, and the scale of the population affected the rate of technological progress due to its effect on: (a) the supply of innovative ideas, (b) the demand for new technologies, (c) the rate of technological diffusion, (d) the division of labour, and (e) the scope for trade.

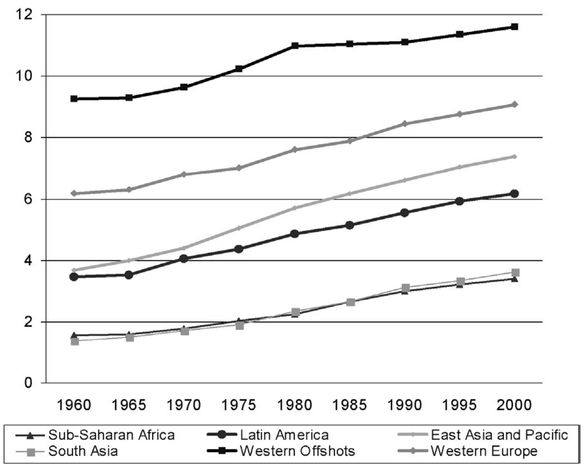

Figure 31. The evolution of average years of education: 1960–2000. Source: Barro and Lee (2000).

children to have.

- Quantity vs quality trade-off depends on the acceleration of TP.

3.1 Setup

The model consists of an overlapping generations economy in which activity extends over infinite discrete time. In every period the economy produces a single homogeneous good using an exogenous amount of land (X) and efficiency units of labour (H_t) as inputs. The number of efficiency units of labour is determined by households' decisions in the preceding period regarding the number and level of human capital of their children.

3.2 Production of final good

The production function has a Cobb-Douglas form $Y_t = H_t^{\alpha}(A_t X)^{1-\alpha}$, where $\alpha \in (0, 1)$ and A_t represents the endogenously determined (land-augmenting) technological level in period t, so $A_t X$ are therefore the "effective resources" employed in production at period t. Defining $h_t = H_t/L_t$ as the level of efficiency units of labour per worker, and $x_t = (A_t X)/L_t$ as the level of effective resources per worker we get:

$$y_t = h_t^{\alpha} x_t^{1-\alpha}$$

We use the price of the final good as the *numeraire*. There are no property rights over land, so the return to land is zero. The market for efficiency units of labour is perfectly competitive, so the wage per efficiency unit of labour is equal to the output per efficiency unit of labour: $w_t = \alpha(x_t/h_t)^{1-\alpha}$.

3.3 Preferences and constraints

People have only one parent and live two periods. In the first period (childhood), they consume a fraction of their parent time. In the second (parenthood), they are endowed with one unit of time, they use part of it in raring children, and the rest in working. They obtain a wage in return for their work that they can spend on their own consumption. Parents decide how many children to have and of which quality, considering that children consume part of their time and their consumption increases with their quality.

Preferences of each individual are defined over consumption, above a subsistence level $\tilde{c} > 0$, i.e. $c_t \geq \tilde{c} \forall t$, and over the quantity and quality of (surviving) children as:

$$u_t = (c_t)^{1-\gamma}(n_t h_{t+1})^{\gamma}$$

where $\gamma \in (0, 1)$, where c_t is the consumption of individual of generation t, n_t is the number of children of each individual at t, and h_{t+1} is the level of human capital of each child. The utility function is strictly monotonically increasing and strictly quasi-concave, satisfying the conventional boundary conditions that assure, for a sufficiently high income, the existence of an interior solution for the utility maximization problem. However, for a sufficiently low level of income the subsistence consumption constraint is binding and there is a corner solution with respect to the consumption level.

Let $\tau + e_{t+1} \leq 1$ be the time cost for a member of generation t of raising a child with a level of education (quality) e_{t+1}. That is, τ is the fraction of the individual's unit time endowment that is required in order to raise a child, regardless of quality, and e_{t+1} is the fraction of the individual's unit time endowment that is devoted for the education of each child.

Consider members of generation t, each is endowed with one unit time they can devote to labour. One unit of labour gives h_t efficiency units of labour at time t. Define potential income, z_t, as the earning if the entire time endowment is devoted to labour force participation, earning the competitive market wage, w_t, per efficiency unit. The potential income, $z_t = w_t h_t 1$, is divided between consumption, c_t, and expenditure on child rearing (quantity as well as quality), evaluated according to the value of the time cost, i.e., $w_t h_t[\tau + e_{t+1}]$, per child. Hence, in the second period of life (parenthood), the individual faces the budget constraint

$$w_t h_t n_t(\tau + e_{t+1}) + c_t \leq w_t h_t = z_t$$

3.4 Production of human capital

Individuals' level of human capital is determined by their quality (education) as well as by the technological environment.

$$h_{t+1} = h(e_{t+1}, g_{t+1})$$

where $g_{t+1} = (A_{t+1} - A_t)/A_t$.

We assume that

$$\begin{aligned} h_e &> 0 \quad h_{ee} < 0 \\ h_g &< 0 \quad h_{gg} > 0 \\ h_{eg} &> 0 \quad h(0,0) = 1 \end{aligned}$$

Technological progress reduces the adaptability of existing human capital for the new technological environment (the 'erosion effect'). Education, however, lessens the adverse effects of technological progress. That is, skilled individuals have a comparative advantage in adapting to the new technological environment. In particular, the time required for learning the new technology diminishes with the level of education and increases with the rate of technological change. Education lessens the adverse effect of technological progress. That is, technology complements skills in the production of human capital. In the absence of investment in quality, each individual has a basic level human capital that is normalized to 1 in a stationary technological environment.

3.5 Household optimization

By choosing how much to invest in each kid and how many of them to have (e_{t+1} and n_t), each parent will be deciding also how much to consume at t. The problem is therefore to find:

$$\{n_t, e_{t+1}\} = argmax\{w_t h_t[1 - n_t(\tau + e_{t+1})]\}^{1-\gamma}\{n_t h(e_{t+1}, g_{t+1})\}^{\gamma}$$

subject to $c_t > \tilde{c}$, and $(n_t, e_{t+1}) > 0$.

There is a threshold of potential income $\tilde{z}$ that makes consumption above subsistence level, i.e. $c_t > \tilde{c}$, possible. It is possible to show that $\tilde{z} = \tilde{c}/(1-\gamma)$.

As long as $z_t > \tilde{z}$ we have an interior solution to the above problem, meaning a fraction of time $1-\gamma$ is spent working to consume, while the remaining γ is used raising children. On the contrary, if $z_t \leq \tilde{z}$, the subsistence constraint is binding, $c_t = \tilde{c}$, so the fraction of time necessary to assure subsistence consumption is larger than $1-\gamma$ and the fraction of time devoted for child rearing is therefore below γ.

So we have:

$$n_t[\tau + e_{t+1}] = \begin{cases} \gamma & \text{if } z_t > \tilde{z} \\ 1 - [\tilde{c}/w_t h_t] & \text{if } z_t \leq \tilde{z} \end{cases}$$

Figure 37 shows that higher potential income levels shift the constraint of the consumer to the right and up. As long as potential income is low the consumer cannot reach the subsistence level of consumption without devoting more than $1-\gamma$ of its time to working. Any increase in z_t implies less time to work, more time to child caring (still below γ) and consumption stuck at $\tilde{c}$. The income expansion path is vertical as long as the subsistence consumption constraint is binding. Once the level of income is sufficiently high such that

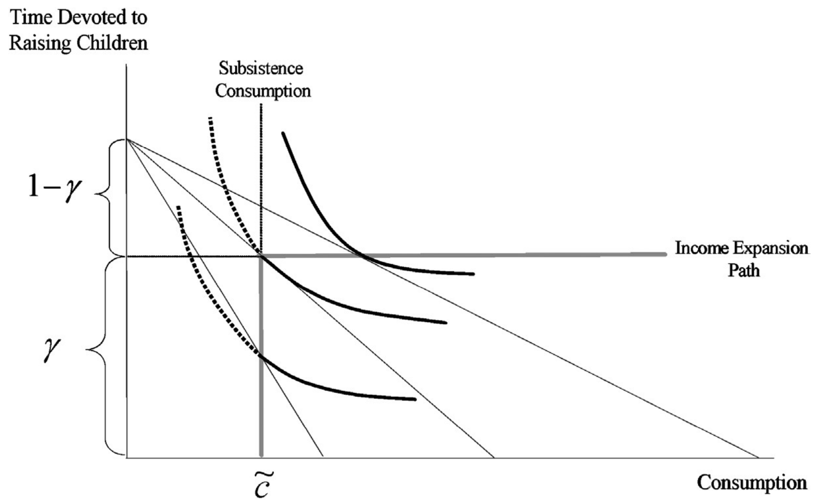

Figure 37. Preferences, constraints, and income expansion path.

the subsistence consumption constraint is not binding, the income expansion path becomes horizontal at a level γ in terms of time devoted to child rearing. Further increases in z_t then imply a constant split of time between activities and more consumption ($c_t > \tilde{c}$).

We can show that the level of education chosen by members of generation t for their children, e_{t+1}, is an increasing function of the growth in technological progress from t to $t+1$, g_{t+1}. Moreover, education is a concave function of g_{t+1} (as long as we assume h_{egg} is low enough).

We can write:

$$e_{t+1} = e(g_{t+1}) \begin{cases} = 0 & \text{if } g_{t+1} \leq \hat{g} \\ > 0 & \text{if } g_{t+1} > \hat{g} \end{cases}$$

with $\hat{g} > 0$, $e' > 0$ and $e'' < 0$ $\forall g_{t+1} > \hat{g} > 0$. In the above equation, $e_{t+1} > 0$ only above a (positive) level of g_{t+1}. This is introduced in the model by assumption.

3.6 Population

Now, what determines the quantity of children? Notice that we can solve for n_t and obtain:

$$n_t = \begin{cases} \frac{\gamma}{\tau+e(g_{t+1})} = n^b(g_{t+1}) & \text{if } z_t > \tilde{z} \\ \frac{1-\tilde{c}/z_t}{\tau+e(g_{t+1})} = n^a(g_{t+1}, z(e_t, g_t, x_t)) & \text{if } z_t \leq \tilde{z} \end{cases}$$

where $z_t = w_t h_t = z(e_t, g_t, x_t)$ and the second equality uses the fact that $w_t = w(x_t, h_t)$ and $h_t = h(e_t, g_t)$. The above equation shows how both the Malthusian and Post-Malthusian dynamics coexist in this framework. The quantity of children will depend only on the future growth rate of technological progress after a certain income threshold is surpassed. Before that, it also depends on the (potential) income level.

The evolution of population is given by $L_{t+1} = n_t L_t$ with $L_0 > 0$ being a parameter. So we obtain:

$$L_{t+1} = \begin{cases} L_t n^b(g_{t+1}) & \text{if } z_t > \tilde{z} \\ L_t n^a(g_{t+1}, z(e_t, g_t, x_t)) & \text{if } z_t \leq \tilde{z} \end{cases}$$

With the above expression we can summarize some of the key mechanisms in place in the model:

1. An increase in the rate of technological progress reduces the number of children and increases their quality:
$$\frac{\partial n_t}{\partial g_{t+1}} \leq 0 \text{ and } \frac{\partial e_{t+1}}{\partial g_{t+1}} \geq 0$$
2. If the subsistence consumption constraint is binding (i.e., if parental potential income is below $\tilde{z}$), an increase in parental potential income raises the number of children, but has no effect on their quality:
$$\frac{\partial n_t}{\partial z_t} > 0 \text{ and } \frac{\partial e_{t+1}}{\partial z_t} = 0 \text{ if } z_t < \tilde{z}$$
3. If the subsistence consumption constraint is not binding (i.e., if parental potential income is above $\tilde{z}$), an increase in parental potential income does not affect the number of children and their quality:
$$\frac{\partial n_t}{\partial z_t} = \frac{\partial e_{t+1}}{\partial z_t} = 0 \text{ if } z_t > \tilde{z}$$

3.7 Technological progress

We assume that technological progress, g_{t+1}, that takes place between periods t and $t+1$, is a function of the education per capita among the working generation in period t, e_t, and the population size in period t, L_t:

$$g_{t+1} = g(e_t, L_t)$$

where with a sufficiently large population size $g(0, L_t) > 0$, $g_i > 0$ and $g_{ii} < 0$ for $i = e, L$. [4] Hence, for a sufficiently large population size, the rate of

[4] For a sufficiently small population the rate of technological progress is strictly positive only every several periods. Furthermore, the number of periods that pass between two episodes of technological improvement declines with the size of population. These assumptions assure that in early stages of development the economy is in a Malthusian steady state with zero growth rate of output per capita, but ultimately the growth rate is positive and slow. If technological progress would occur in every time period at a pace that increases with the size of population, the growth rate of output per capita would always be positive, despite the adjustment in the size of population.

technological progress between time t and $t+1$ is a positive, increasing, strictly concave function of the size of adult population and the level of education of the working generation at time t . Furthermore, the rate of technological progress is positive even if labour quality is zero.

3.8 Effective resources

The evolution of effective resources per worker, $x_t = (A_t X)/L_t$, is determined by the evolution of population and technology. The level of effective resources per worker in period $t+1$ is

$$x_{t+1} = \frac{1+g_{t+1}}{n_t} x_t$$

So using previous results we can write:

$$x_t = \begin{cases} \frac{[\tau + e(g(e_t, L_t))][1+g(e_t,L_t)]}{\gamma} x_t = \phi^b(e_t, L_t) x_t & \text{if } z_t > \tilde{z} \\ \frac{[\tau + e(g(e_t, L_t))][1+g(e_t,L_t)]}{1-\tilde{c}/z_t} x_t = \phi^a(e_t, g_t, x_t, L_t) x_t & \text{if } z_t \leq \tilde{z} \end{cases}$$

where $\phi_e^b > 0$ and $\phi_x^a < 0 \ \forall e_t > 0$.

3.9 The dynamic system

The equilibrium path of the economy is given by a sequence $\{e_t, g_t, x_t, L_t\}_{t=0}^{\infty}$ that satisfies the above equations describing the joint evolution of education, technological progress, effective resources per capita, and population over time. As is clear, the dynamic system is characterized by two regimes, the Malthusian and Post-Malthusian regimes separated by potential resource level $\tilde{z}$.

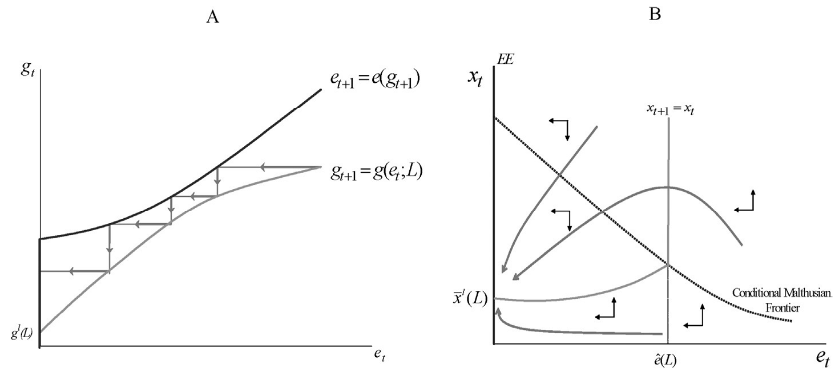

Figure 38. The evolution of technology, g_t, education, e_t, and effective resources, x_t: small population.

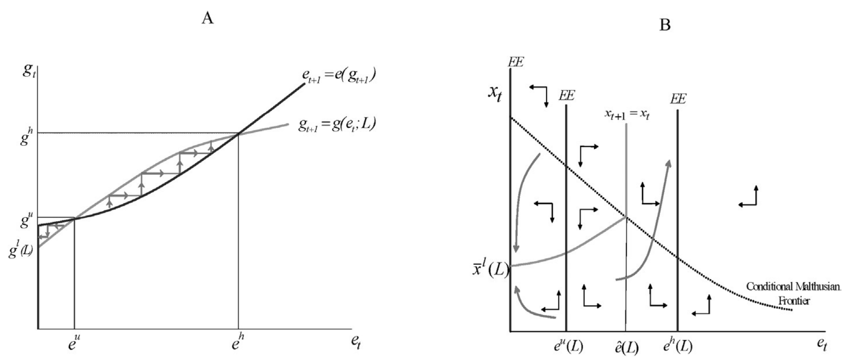

Figure 39. The evolution of technology, g_t, education, e_t, and effective resources, x_t: moderate population.

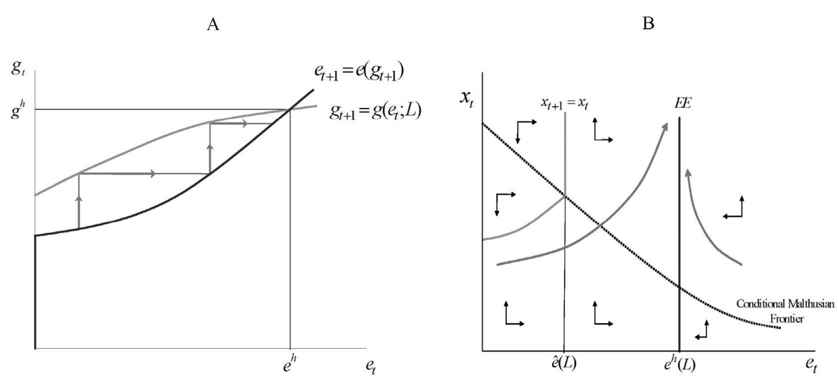

Figure 40. The evolution of technology, g_t, education, e_t, and effective resources, x_t: large population.

In both regimes, however, the analysis of the dynamic system is greatly simplified by the fact that the evolution of e_t and g_t are independent of whether the subsistence constraint is binding, and by the fact that, for a given population size L, the joint evolution of e_t and g_t is determined independently of x_t. The education level of workers in period $t+1$ depends only on the level of technological progress expected between period t and period $t+1$, while given L, technological progress between periods t and $t+1$ depends only on the level of education of workers in period t. Thus, for a given population size L, the dynamics of technology and education can be analysed independently of the evolution of resources per capita.

In light of the properties of the functions $e(g_{t+1})$ and $g(e_t; L)$ this dynamical subsystem is characterized by three qualitatively different configurations, which are depicted in Figures 38A, 39A and 40A. The economy shifts endogenously from one configuration to another as population increases and the curve $g(e_t; L)$ shifts upward to account for the effect of an increase in population.

In Figure 38A, for a range of small population size, the dynamical system is characterized by globally stable steady-state equilibria, $(\bar{e}(L), \bar{g}(L)) = (0, g^l(L))$.

In Figure 39A, for a range of moderate population size, the dynamical system is characterized by three steady-state equilibria, two locally stable steady-state equilibria: $(\bar{e}(L), \bar{g}(L)) = (0, g^l(L))$ and $(\bar{e}(L), \bar{g}(L)) = (e^h(L), g^h(L))$, and an interior unstable steady state $(\bar{e}(L), \bar{g}(L)) = (e^u(L), g^u(L))$.

Finally, in Figure 40A, for a range of large population sizes, the dynamical system is characterized by globally stable steady-state equilibria, $(\bar{e}(L), \bar{g}(L)) = (e^h(L), g^h(L))$.

To analyse the figures in the right hand side which give the dynamics of the rest of the variables, we need to introduce the following definitions.

Let the **Conditional Malthusian Frontier** be the set of all pairs $(e_t, x_t; L)$ for which, conditional on a given technological level g_t, individuals incomes equal $\tilde{z}$. Following the definitions of z_t and $\tilde{z}$, we can define: $MM|_{g_t} = \{(e_t, x_t; L) : x_t^{1-\alpha} h(e_t, g_t)^\alpha = \tilde{c}/(1-\gamma)|g_t\}$. Notice that, conditional on the technological progress, this locus describes a unique relationship between x_t and e_t. In fact, x_t is a decreasing strictly convex function of e_t along the $MM|_{g_t}$ locus. Furthermore, it intersects the x_t axis and approaches asymptotically the e_t axis as x_t approaches infinity. The frontier shifts upward as g_t increases in the process of development.

Let **XX** be the locus of all triplets $(e_t, x_t, g_t; L)$ such that the effective resources per worker, x_t, is at steady state: $XX = \{(e_t, x_t, g_t; L) : x_{t+1} = x_t\}$. Along the XX locus we must have $n_t = 1 + g_{t+1}$, so the growth rates of effective resources and technology must be equal. How does the XX-locus look like? Above the Malthusian frontier, people's potential income surpasses survival, so there is enough time to childrearing. The fraction of time devoted to childrearing is independent of the level of effective resources per worker. In this case, the growth rate of population will just be a negative function of the growth rate of technology, since for higher technology growth, parents will spend more of their resources on child quality and thus less on child quantity.

Thus there will be a particular level of technological progress which induces a rate of population growth that keeps effective resources constant. Since the growth rate of technology is, in turn, a positive function of the level of education, this rate of technology growth will correspond to a particular level of education, denoted $\tilde{e}(L)$. Below the Malthusian Frontier, the growth rate of population depends on the level of effective resources per capita, x, as well as on the growth rate of technology.

The lower is x, the smaller the fraction of time devoted to child-rearing, and so the lower is population growth n. Thus, below the Malthusian frontier, a lower value of effective resources per capita would imply that lower values of technology growth (and thus education) would be consistent with effective resources being constant. Thus, as drawn in Figures B, lower values of x are associated with lower values of e on the part of the XX locus that is below the Malthusian frontier.

Let **EE** be the locus of all triplets $(e_t, x_t, g_t; L)$ such that the quality of labour, e_t, is in a steady state: $EE = \{(e_t, x_t, g_t; L) : e_{t+1} = e_t\}$. As follows from our previous results, $e_{t+1} = e(g(e_t; L))$ and thus, for a given population size, the steady-state values of e_t are independent of the values of x_t and g_t. The locus EE evolves through three phases in the process of development, corresponding to the three phases that describe the evolution of education and technology, as depicted in Figures A.

In early stages of development, when population size is sufficiently small, the joint evolution of education and technology is characterized by a globally stable temporary steady-state equilibrium, $(\bar{e}(L), \bar{g}(L)) = (0, g^l(L))$, as depicted in Figure 38A. The corresponding EE locus, depicted in the space $(e_t, x_t; L)$ in Figure 38B, is vertical at the level $e = 0$, for a range of small population sizes. Furthermore, for this range, the global dynamics of e_t are given by:

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = 0 \\ < 0 & \text{if } e_t > 0 \end{cases}$$

In later stages of development as population size increases sufficiently, the joint evolution of education and technology is characterized by multiple locally stable temporary steady-state equilibria, as depicted in Figure 39A. The corresponding EE locus, depicted in the space $(e_t, x_t; L)$ in Figure 39B, consists of three vertical lines corresponding to the three steady-state equilibria for the value of e_t . That is, $e = 0$, $e = e^u(L)$, and $e = e^h(L)$. The vertical line $e = e^u(L)$ shifts leftward, and $e = e^h(L)$ shifts rightward as population size increases. Furthermore, the global dynamics of e_t in this configuration are given by:

$$e_{t+1} - e_t \begin{cases} < 0 & \text{if } 0 < e_t < e^u(L) \text{ or } e_t > e^h(L) \\ = 0 & \text{if } e_t = (0, e^u(L), e^h(L)) \\ > 0 & \text{if } e^u(L) < e_t < e^h(L) \end{cases}$$

In mature stages of development when population size is sufficiently large, the joint evolution of education and technology is characterized by a globally stable steady-state equilibrium, $(\bar{e}(L), \bar{g}(L)) = (e^h(L), g^h(L))$, as depicted in

Figure 40A. The corresponding EE locus, as depicted in Figure 40B in the space $(e_t, x_t; L)$, is vertical at the level $e = e^h(L)$. This vertical line shifts rightward as population size increases. Furthermore, the global dynamics of e_t in this configuration are given by:

$$e_{t+1} - e_t \begin{cases} < 0 & \text{if } 0 \geq e_t < e^h(L) \\ = 0 & \text{if } e_t = e^h(L) \\ > 0 & \text{if } e^h(L) < e_t \end{cases}$$

Having understood the dynamics of e, let us now look at the dynamics of x in the B figures, so let us analyse the conditional steady-state equilibria. In early stages of development, when population size is sufficiently small, the dynamic system, as depicted in Figure 38B, is characterized by a unique and globally stable conditional steady-state equilibrium. It is given by a point of intersection between the EE locus and the XX locus. That is, conditional on a given technological level, g_t, the Malthusian steady state $(0, \bar{x}(L))$ is globally stable. In later stages of development as population size increases sufficiently, the dynamic system as depicted in Figure 39B is characterized by two conditional steady-state equilibria. The Malthusian conditional steady-state equilibrium is locally stable, whereas the steady-state equilibrium $(e^u(L), x^u(L))$ is a saddle point. [5] For education levels above $e^u(L)$ the system converges to a stationary level of education $e^h(L)$ and possibly to a steady-state growth rate of x_t. In mature stages of development when population size is sufficiently large, the system convergences globally to an educational level $e^h(L)$ and possibly to a steady-state growth rate of x_t.

3.10 Main Predictions of the model

The following are the main predictions of the model that are supported by the empirical evidence:

1. During the Malthusian epoch the growth rate of output per capita is nearly zero and the growth rate of population is minuscule, reflecting the sluggish pace of technological progress and the full adjustment of population to the expansion of resources.

2. The reinforcing interaction between population and technology during the Malthusian epoch increased the size of the population sufficiently so as to support a faster pace of technological progress, generating the transition to the Post-Malthusian Regime. The growth rates of output per capita increased significantly, but the positive Malthusian effect of income per capita on population growth was still maintained, generating a sizeable increase in population growth, and offsetting some of the potential gains in income per capita. Moreover, human capital accumulation did not play

[5] Convergence to the saddle point takes place only if the level of education is e^u. That is, the saddle path is the entire vertical line that corresponds to $e_t = e^u$.

a significant role in the transition to the Post-Malthusian Regime and thus in the early take-off in the first phase of the Industrial Revolution.

3. The acceleration in the rate of technological progress increased the industrial demand for human capital in the later part of Post-Malthusian Regime (i.e., the second phase of industrialization), inducing significant investment in human capital, and triggering the demographic transition and a rapid pace of economic growth.