CHAPTER

TEN

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The Nth-derivative test developed in the preceding chapter equips us for the task of locating the extreme values of any objective function, as long as it involves only one choice variable, possesses derivatives to the desired order, and sooner or later yields a nonzero derivative value at the critical value x_0. In the examples cited in Chap. 9, however, we made use only of polynomial and rational functions, for which we know how to obtain the necessary derivatives. Suppose that our objective function happened to be an *exponential* one, such as

$$y = 8^{x - \sqrt{x}}$$

Then we are still helpless in applying the derivative criterion, because we have yet to learn how to differentiate such a function. This is what we shall do in the present chapter.

Exponential functions, as well as the closely related logarithmic functions, have important applications in economics, especially in connection with growth problems, and in economic dynamics in general. The particular application relevant to the present part of the book, however, involves a class of optimization problems in which the choice variable is *time*. For example, a certain wine dealer may have a stock of wine, the market value of which, owing to its vintage year, is known to increase with time in some prescribed fashion. The problem is to determine the best time to sell that stock on the basis of the wine-value function, after taking into account the interest cost involved in having the money capital tied up in that stock. Exponential functions may enter into such a problem in two ways. In the first place, the value of the wine may increase with time according to some *exponential law of growth*. In that event, we would have an exponential wine-value function. This is only a possibility, of course, and not a certainty. When we give consideration to the interest cost, however, a *sure* entry is provided for an exponential function because of the fact of interest compounding, which will be explained presently. Thus we must study the nature of exponential functions before we can discuss this type of optimization problem.

Since our primary purpose is to deal with time as a choice variable, let us now switch to the symbol t—in lieu of x—to indicate the independent variable in the subsequent discussion. (However, this same symbol t can very well represent variables other than time also.)

10.1 THE NATURE OF EXPONENTIAL FUNCTIONS

As introduced in connection with polynomial functions, the term *exponent* means an indicator of the power to which a variable is to be raised. In power expressions such as x^3 or x^5, the exponents are *constants*; but there is no reason why we cannot also have a *variable* exponent, such as in 3^x or 3^t, where the number 3 is to be raised to varying powers (various values of x). A function whose *independent* variable appears in the role of an exponent is called an *exponential function*.

Simple Exponential Function

In its simple version, the exponential function may be represented in the form

$$(10.1) \qquad y = f(t) = b^t \qquad (b > 1)$$

where y and t are the dependent and independent variables, respectively, and b denotes a fixed *base* of the exponent. The domain of such a function is the set of all real numbers. Thus, unlike the exponents in a polynomial function, the variable exponent t in (10.1) is not limited to positive integers—unless we wish to impose such a restriction.

But why the restriction of $b > 1$? The explanation is as follows. In view of the fact that the domain of the function in (10.1) consists of the set of all real numbers, it is possible for t to take a value such as $\frac{1}{2}$. If b is allowed to be negative, the half power of b will involve taking the square root of a negative number. While this is not an impossible task, we would certainly prefer to take the easy way out by restricting b to be positive. Once we adopt the restriction $b > 0$, however, we might as well go all the way to the restriction $b > 1$: The restriction $b > 1$ differs from $b > 0$ only in the further exclusion of the cases of (1) $0 < b < 1$ and (2) $b = 1$; but as will be shown, the first case can be subsumed under the restriction $b > 1$, whereas the second case can be dismissed outright. Consider the first case. If $b = \frac{1}{5}$, then we have

$$y = \left(\frac{1}{5}\right)^t = \frac{1}{5^t} = 5^{-t}$$

This shows that a function with a fractional base can easily be rewritten into one with a base greater than 1. As for the second case, the fact that $b = 1$ will give us the function $y = 1^t = 1$, so that the exponential function actually degenerates into a constant function; it may therefore be disqualified as a member of the exponential family.

Graphical Form

The graph of the exponential function in (10.1) takes the general shape of the curve in Fig. 10.1. The curve drawn is based on the value $b = 2$; but even for other values of b, the same general configuration will prevail.

Several salient features of this type of exponential curve may be noted. First, it is continuous and smooth everywhere; thus the function should be everywhere differentiable. As a matter of fact, it is continuously differentiable any number of times. Second, it is monotonically increasing, and in fact y increases at an increasing rate throughout. Consequently, both the first and second derivatives of the function $y = b^t$ should be positive—a fact we should be able to confirm after we have developed the relevant differentiation formulas. Third, we note that, even though the domain of the function contains negative as well as positive numbers, the range of the function is limited to the open interval $(0, \infty)$. That is, the dependent variable y is invariably *positive*, regardless of the sign of the independent variable t.

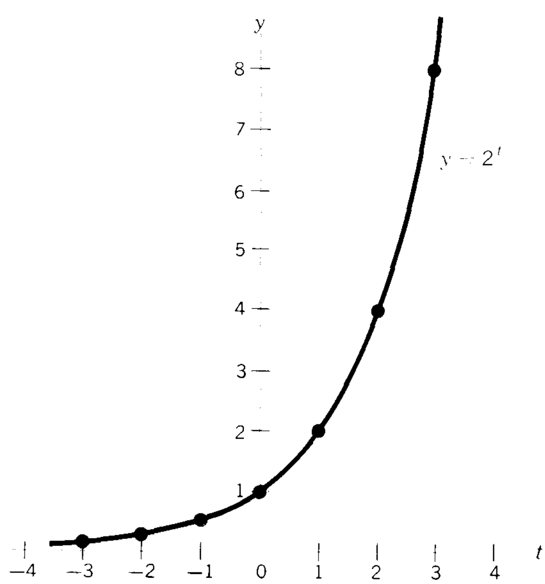

Figure 10.1

The monotonicity of the exponential function entails at least two interesting and significant implications. First, we may infer that the exponential function must have an inverse function, which is itself monotonic. This inverse function, we shall find, turns out to be a *logarithmic* function. Second, since monotonicity means that there is a unique value of t for a given value of y and since the range of the exponential function is the interval $(0, \infty)$, it follows that we should be able to express *any positive number* as a unique power of a base $b > 1$. This can be seen from Fig. 10.1, where the curve of $y = 2^t$ covers all the positive values of y in its range; therefore any positive value of y must be expressible as some unique power of the number 2. Actually, even if the base is changed to some other real number greater than 1, the same range holds, so that it is possible to express any positive number y as a power of any base $b > 1$.

Generalized Exponential Function

This last point deserves closer scrutiny. If a positive y can indeed be expressed as powers of various alternative bases, then there must exist a general procedure of *base conversion*. In the case of the function $y = 9^t$, for instance, we can readily transform it into $y = (3^2)^t = 3^{2t}$, thereby converting the base from 9 to 3, provided the exponent is duly altered from t to $2t$. This change in exponent, necessitated by the base conversion, does not create any new type of function, for, if we let $w = 2t$, then $y = 3^{2t} = 3^w$ is still in the form of (10.1). From the point of view of the base 3, however, the exponent is now $2t$ rather than t. What is the effect of adding a numerical coefficient (here, 2) to the exponent t?

The answer is to be found in Fig. 10.2a, where two curves are drawn—one for the function $y = f(t) = b^t$ and one for another function $y = g(t) = b^{2t}$. Since the exponent in the latter is exactly twice that of the former, and since the identical base is adopted for the two functions, the assignment of an arbitrary value $t = t_0$ in the function g and $t = 2t_0$ in the function f must yield the same value:

$$f(2t_0) = g(t_0) = b^{2t_0} = y_0$$

Thus the distance y_0J will be half of y_0K. By similar reasoning, for any value of y, the function g should be exactly halfway between the function f and the vertical axis. It may be concluded, therefore, that the *doubling* of the exponent has the effect of compressing the exponential curve *halfway* toward the y axis, whereas *halving* the exponent will extend the curve away from the y axis to *twice* the horizontal distance.

It is of interest that both functions share the same vertical intercept

$$f(0) = g(0) = b^0 = 1$$

The change of the exponent t to $2t$, or to any other multiple of t, will leave the vertical intercept unaffected. In terms of *compressing*, this is because compressing a zero horizontal distance will still yield a zero distance.

The change of exponent is one way of modifying—and generalizing—the exponential function of (10.1); another is to attach a coefficient to b^t, such as $2b^t$. [*Warning*: $2b^t \neq (2b)^t$.] The effect of such a coefficient is also to compress or extend the curve, except that this time the direction is vertical. In Fig. 10.2*b*, the higher curve represents $y = 2b^t$, and the lower one is $y = b^t$. For every value of t, the former must obviously be twice as high, because it has a y value twice as large as the latter. Thus we have $t_0J' = J'K'$. Note that the vertical intercept, too, is changed in the present case. We may conclude that *doubling* the coefficient (here, from 1 to 2) serves to extend the curve away from the horizontal axis to *twice* the vertical distance, whereas *halving* the coefficient will compress the curve *halfway* toward the t axis.

With the knowledge of the two modifications discussed above, the exponential function $y = b^t$ can now be generalized to the form

$$y = ab^{ct} \tag{10.2}$$

where a and c are "compressing" or "extending" agents. When assigned various values, they will alter the position of the exponential curve, thus generating a whole family of exponential curves (functions). If a and c are positive, the general configuration shown in Fig. 10.2 will prevail; if a or c or both are *negative*, however, then fundamental modifications will occur in the configuration of the curve (see Exercise 10.1-5 below).

A Preferred Base

What prompted the discussion of the change of exponent from t to ct was the question of base conversion. But, granting the feasibility of base conversion, why

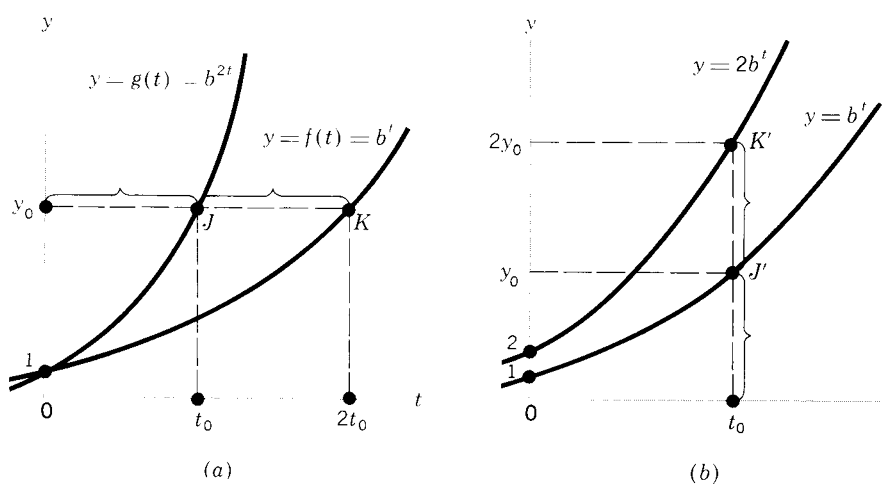

Figure 10.2

would one want to do it anyhow? One answer is that some bases are more convenient than others as far as mathematical manipulations are concerned.

Curiously enough, in calculus, the preferred base happens to be a certain irrational number denoted by the symbol e:

$$e = 2.71828\ldots$$

When this base e is used in an exponential function, it is referred to as a *natural exponential function*, examples of which are

$$y = e^t \qquad y = e^{3t} \qquad y = Ae^{rt}$$

These illustrative functions can also be expressed by the alternative notations

$$y = \exp(t) \qquad y = \exp(3t) \qquad y = A\exp(rt)$$

where the abbreviation exp (for exponential) indicates that e is to have as its exponent the expression in parentheses.

The choice of such an outlandish number as $e = 2.71828\ldots$ as the preferred base will no doubt seem bewildering. But there is an excellent reason for this choice, for the function e^t possesses the remarkable property of being its own derivative! That is,

$$\frac{d}{dt}e^t = e^t$$

a fact which will reduce the work of differentiation to practically no work at all. Moreover, armed with this differentiation rule—to be proved later in this chapter—it will also be easy to find the derivative of a more complicated natural exponential function such as $y = Ae^{rt}$. To do this, first let $w = rt$, so that the function becomes

$$y = Ae^w \qquad \text{where } w = rt\text{, and } A, r \text{ are constants}$$

Then, by the chain rule, we can write

$$\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = Ae^w(r) = rAe^{rt}$$

That is,

$$\frac{d}{dt}Ae^{rt} = rAe^{rt} \qquad (10.3)$$

The mathematical convenience of the base e should thus be amply clear.

EXERCISE 10.1

1 Plot in a single diagram the graphs of the exponential functions $y = 3^t$ and $y = 3^{2t}$.

(*a*) Do the two graphs display the same general positional relationship as shown in Fig. 10.2*a*?

(*b*) Do these two curves share the same y intercept? Why?

(*c*) Sketch the graph of the function $y = 3^{3t}$ in the same diagram.

2 Plot in a single diagram the graphs of the exponential functions $y = 4^t$ and $y = 3(4^t)$.

(a) Do the two graphs display the general positional relationship suggested in Fig. 10.2b?

(b) Do the two curves have the same y intercept? Why?

(c) Sketch the graph of the function $y = \frac{3}{2}(4^t)$ in the same diagram.

3 Taking for granted that e^t is its own derivative, use the chain rule to find dy/dt for the following:

(a) $y = e^{5t}$ (b) $y = 4e^{3t}$ (c) $y = 6e^{2t}$

4 In view of our discussion about (10.1), do you expect the function $y = e^t$ to be monotonically increasing at an increasing rate? Verify your answer by determining the signs of the first and second derivatives of this function. In doing so, remember that the domain of this function is the set of all real numbers, i.e., the interval $(-\infty, \infty)$.

5 In (10.2), if negative values are assigned to a and c, the general shape of the curves in Fig. 10.2 will no longer prevail. Examine the change in curve configuration by contrasting (a) the case of $a = -1$ against the case of $a = 1$, and (b) the case of $c = -1$ against the case of $c = 1$.

10.2 NATURAL EXPONENTIAL FUNCTIONS AND THE PROBLEM OF GROWTH

The pertinent questions still unanswered are: How is the number e defined? Does it have any economic meaning in addition to its mathematical significance as a convenient base? And, in what ways do natural exponential functions apply to economic analysis?

The Number e

Let us consider the following function:

$$f(m) = \left(1 + \frac{1}{m}\right)^m \qquad (10.4)$$

If larger and larger values are assigned to m, then $f(m)$ will also assume larger values; specifically, we find that

$$f(1) = \left(1 + \tfrac{1}{1}\right)^1 = 2$$

$$f(2) = \left(1 + \tfrac{1}{2}\right)^2 = 2.25$$

$$f(3) = \left(1 + \tfrac{1}{3}\right)^3 = 2.37037\ldots$$

$$f(4) = \left(1 + \tfrac{1}{4}\right)^4 = 2.44141\ldots$$

$$\vdots$$

Moreover, if m is increased indefinitely, then $f(m)$ will converge to the number

$2.71828\ldots \equiv e$; thus e may be defined as the limit of (10.4) as $m \to \infty$:

$$e \equiv \lim_{m\to\infty} f(m) = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m \tag{10.5}$$

That the approximate value of e is 2.71828 can be verified by finding the Maclaurin series of the function $\phi(x) = e^x$—with x used here to facilitate the application of the expansion formula (9.14). Such a series will give us a polynomial approximation to e^x, and thus the value of e ($= e^1$) may be approximated by setting $x = 1$ in that polynomial. If the remainder term R_n approaches zero as the number of terms in the series is increased indefinitely, i.e., if the series is convergent to $\phi(x)$, then we can indeed approximate the value of e to any desired degree of accuracy by making the number of included terms sufficiently large.

To this end, we need to have derivatives of various orders for the function. Accepting the fact that the first derivative of e^x is e^x itself, we can see that the derivative of $\phi(x)$ is simply e^x and, similarly, that the second, third, or any higher-order derivatives must be e^x as well. Hence, when we evaluate all the derivatives at the expansion point ($x_0 = 0$), we have the gratifyingly neat result

$$\phi'(0) = \phi''(0) = \cdots = \phi^{(n)}(0) = e^0 = 1$$

Consequently, by setting $x_0 = 0$ in (9.14), the Maclaurin series of e^x is

$$\begin{aligned} e^x = \phi(x) &= \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \cdots \\ &\qquad + \frac{\phi^{(n)}(0)}{n!}x^n + R_n \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n \end{aligned}$$

The remainder term R_n, according to (9.15), can be written as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{e^p}{(n+1)!}x^{n+1}$$

$$\left[\phi^{(n+1)}(x) = e^x;\ \therefore \phi^{(n+1)}(p) = e^p\right]$$

Inasmuch as the factorial expression $(n + 1)!$ will increase in value more rapidly than the power expression x^{n+1} (for a finite x) as n increases, it follows that $R_n \to 0$ as $n \to \infty$. Thus the Maclaurin series converges, and the value of e^x may, as a result, be expressed as an *infinite series*—an expression involving an infinite number ($n \to \infty$) of additive terms which follow a consistent, recognizable pattern of formation, and in which the remainder term R_n disappears ($R_n \to 0$):

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \tag{10.6}$$

As a special case, for $x = 1$, we find that

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \\ &= 2 + 0.5 + 0.1666667 + 0.0416667 + 0.0083333 + 0.0013889 \\ &\qquad + 0.0001984 + 0.0000248 + 0.0000028 + 0.0000003 + \cdots \\ &= 2.7182819 \end{aligned}$$

Thus, if we want a figure accurate to five decimal places, we can write $e = 2.71828$. Note that we need not worry about the subsequent terms in the infinite series, because they will be of negligible magnitude if we are concerned only with five decimal places.

An Economic Interpretation of e

Mathematically, the number e is the limit expression in (10.5). But does it also possess some economic meaning? The answer is that it can be interpreted as the result of a special process of interest compounding.

Suppose that, starting out with a principal (or capital) of \$1, we find a hypothetical banker to offer us the unusual interest rate of 100 percent per annum (\$1 interest per year). If interest is to be compounded once a year, the value of our asset at the end of the year will be \$2; we shall denote this value by $V(1)$, where the number in parentheses indicates the frequency of compounding within 1 year:

$$\begin{aligned} V(1) &= \text{initial principal } (1 + \text{interest rate}) \\ &= 1(1 + 100\%) = \left(1 + \tfrac{1}{1}\right)^1 = 2 \end{aligned}$$

If interest is compounded semiannually, however, an interest amounting to 50 percent (half of 100 percent) of principal will accrue at the end of 6 months. We shall therefore have \$1.50 as the new principal during the second 6-month period, in which interest will be calculated at 50 percent of \$1.50. Thus our year-end asset value will be $1.50(1 + 50\%)$; that is,

$$V(2) = (1 + 50\%)(1 + 50\%) = \left(1 + \tfrac{1}{2}\right)^2$$

By analogous reasoning, we can write $V(3) = (1 + \frac{1}{3})^3$, $V(4) = (1 + \frac{1}{4})^4$, etc.; or, in general,

$$(10.7) \qquad V(m) = \left(1 + \frac{1}{m}\right)^m$$

where m represents the frequency of compounding in 1 year.

In the limiting case, when interest is compounded *continuously* during the year, i.e., when m becomes infinite, the value of the asset will grow in a "snowballing" fashion, becoming at the end of 1 year

$$\lim_{m \to \infty} V(m) = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e(\text{dollars}) \qquad [\text{by } (10.5)]$$

Thus, the number $e = 2.71828$ can be interpreted as the year-end value to which a principal of \$1 will grow if interest at the rate of 100 percent per annum is compounded continuously.

Note that the interest rate of 100 percent is only a *nominal interest rate*, for if \$1 becomes $\$e = \$2.718$ after 1 year, the *effective interest rate* is in this case approximately 172 percent per annum.

Interest Compounding and the Function Ae^{rt}

The continuous interest-compounding process just discussed can be generalized in three directions, to allow for: (1) more years of compounding, (2) a principal other than \$1, and (3) a nominal interest rate other than 100 percent.

If a principal of \$1 becomes $\$e$ after 1 year of continuous compounding and if we let $\$e$ be the new principal in the second year (during which every dollar will again grow into $\$e$), our asset value at the end of 2 years will obviously become $\$e(e) = \$\,e^2$. By the same token, it will become $\$e^3$ at the end of 3 years or, more generally, will become $\$e^t$ after t years.

Next, let us change the principal from \$1 to an unspecified amount, $\$A$. This change is easily taken care of: if \$1 will grow into $\$e^t$ after t years of continuous compounding at the nominal rate of 100 percent per annum, it stands to reason that $\$A$ will grow into $\$Ae^t$.

How about a nominal interest rate of other than 100 percent, for instance, $r = 0.05$ ($= 5$ percent)? The effect of this rate change is to alter the expression Ae^t to Ae^{rt}, as can be verified from the following. With an initial principal of $\$A$, to be invested for t years at a nominal interest rate r, the compound-interest formula (10.7) must be modified to the form

$$V(m) = A\left(1 + \frac{r}{m}\right)^{mt} \tag{10.8}$$

The insertion of the coefficient A reflects the change of principal from the previous level of \$1. The quotient expression r/m means that, in each of the m compounding periods in a year, only $1/m$ of the nominal rate r will actually be applicable. Finally, the exponent mt tells us that, since interest is to be compounded m times a year, there should be a total of mt compoundings in t years.

The formula (10.8) can be transformed into an alternative form

$$\begin{aligned} V(m) &= A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^{rt} \\ &= A\left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt} \qquad \text{where } w \equiv \frac{m}{r} \end{aligned} \tag{10.8'}$$

As the frequency of compounding m is increased, the newly created variable w must increase pari passu; thus, as $m \to \infty$, we have $w \to \infty$, and the bracketed expression in (10.8'), by virtue of (10.5), tends to the number e. Consequently, we find the asset value in the generalized continuous-compounding process to be

$$(10.8'') \qquad V \equiv \lim_{m \to \infty} V(m) = Ae^{rt}$$

as anticipated above.

Note that, in (10.8), t is a *discrete* (as against a *continuous*) variable: it can only take values that are integral multiples of $1/m$. For example, if $m = 4$ (compounding on a quarterly basis), then t can only take the values of $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, etc., indicating that $V(m)$ will assume a new value only at the end of each new quarter. When $m \to \infty$, as in (10.8″), however, $1/m$ will become infinitesimal, and accordingly the variable t will become continuous. In that case, it becomes legitimate to speak of fractions of a year and to let t be, say, 1.2 or 2.35.

The upshot is that the expressions e, e^t, Ae^t, and Ae^{rt} can all be interpreted economically in connection with continuous interest compounding, as summarized in Table 10.1.

Instantaneous Rate of Growth

It should be pointed out, however, that interest compounding is an illustrative, but not exclusive, interpretation of the natural exponential function Ae^{rt}. Interest compounding merely exemplifies the general process of *exponential growth* (here, the growth of a sum of money capital over time), and we can apply the function equally well to the growth of population, wealth, or real capital.

Applied to some context other than interest compounding, the coefficient r in Ae^{rt} no longer denotes the nominal interest rate. What economic meaning does it then take? The answer is that r can be reinterpreted as the *instantaneous rate of growth* of the function Ae^{rt}. (In fact, this is why we have adopted the symbol r, for rate of growth, in the first place.) Given the function $V = Ae^{rt}$, which gives the value of V at each point of time t, the rate of change of V is to be found in the derivative

$$\frac{dV}{dt} = rAe^{rt} = rV \qquad [\text{see (10.3)}]$$

But the *rate of growth* of V is simply the *rate of change* in V expressed in relative (percentage) terms, i.e., expressed as a ratio to the value of V itself. Thus, for any

Table 10.1 Continuous interest compounding

Principal, \$	Nominal interest rate	Years of continuous compounding	Asset value, at the end of compounding process, \$
1	100% (= 1)	1	e
1	100%	t	e^t
A	100%	t	Ae^t
A	r	t	Ae^{rt}

given point of time, we have

$$(10.9) \qquad \text{Rate of growth of } V \equiv \frac{dV/dt}{V} = \frac{rV}{V} = r$$

as was stated above.

Several observations should be made about this rate of growth. But, first, let us clarify a fundamental point regarding the concept of time, namely, the distinction between a *point* of time and a *period* of time. The variable V (denoting a sum of money, or the size of population, etc.) is a *stock* concept, which is concerned with the question: How much of it *exists* at a given moment? As such, V is related to the *point* concept of time; at each point of time, V takes a unique value. The change in V, on the other hand, represents a *flow*, which involves the question: How much of it *takes place* during a given time span? Hence a change in V and, by the same token, the rate of change of V must have reference to some specified period of time, say, per year.

With this understanding, let us return to (10.9) for some comments:

1. The rate of growth defined in (10.9) is an *instantaneous* rate of growth. Since the derivative $dV/dt = rAe^{rt}$ takes a different value at a different point of t, as will $V = Ae^{rt}$, their ratio must also have reference to a specific point (or *instant*) of t. In this sense, the rate of growth is instantaneous.
2. In the present case, however, the instantaneous rate of growth happens to be a constant r, with the rate of growth thus remaining uniform at all points of time. This many not, of course, be true of all growth situations actually encountered.
3. Even though the rate of growth r is measured instantaneously, as of a particular point of time, its magnitude nevertheless has the connotation of so many percent *per unit of time*, say, per year (if t is measured in year units). Growth, by its very nature, can occur only over a time interval. This is why a single still picture (recording the situation at one instant) could never portray, say, the growth of a child, whereas two still pictures taken at different times —say, a year apart—can accomplish this. To say that V has a rate of growth of r at the instant $t = t_0$, therefore, really means that, if the rate r prevailing at $t = t_0$ is allowed to continue undisturbed for one whole unit of time (1 year), then V will have grown by the amount rV at the end of the year.
4. For the exponential function $V = Ae^{rt}$, the *percentage rate* of growth is constant at all points of t, but the *absolute amount* of increment of V increases as time goes on, because the percentage rate will be calculated on larger and larger bases.

Upon interpreting r as the instantaneous rate of growth, it is clear that little effort will henceforth be required to find the rate of growth of a natural exponential function of the form $y = Ae^{rt}$, provided r is a constant. Given a function $y = 75e^{0.02t}$, for instance, we can immediately read off the rate of growth of y as 0.02 or 2 percent per period.

Continuous versus Discrete Growth

The above discussion, though analytically interesting, is still open to question insofar as economic relevance is concerned, because in actuality growth does not always take place on a *continuous* basis—not even in interest compounding. Fortunately, however, even for cases of *discrete* growth, where changes occur only once per period rather than from instant to instant, the continuous exponential growth function can be justifiably used.

For one thing, in cases where the frequency of compounding is relatively high, though not infinite, the continuous pattern of growth may be regarded as an approximation to the true growth pattern. But, more importantly, we can show that a problem of discrete or discontinuous growth can always be transformed into an equivalent continuous version.

Suppose that we have a geometric pattern of growth (say, the *discrete* compounding of interest) as shown by the following sequence:

$$A,\ A(1+i),\ A(1+i)^2,\ A(1+i)^3, \dots$$

where the effective interest rate per period is denoted by i and where the exponent of the expression $(1+i)$ denotes the number of periods covered in the compounding. If we consider $(1+i)$ to be the base b in an exponential expression, then the above sequence may be summarized by the exponential function Ab^t—except that, because of the discrete nature of the problem, t is restricted to integer values only. Moreover, $b = 1 + i$ is a positive number (positive even if i is a *negative* interest rate, say, -0.04), so that it can always be expressed as a power of any real number greater than 1, including e. This means that there must exist a number r such that*

$$1 + i = b = e^r$$

Thus we can transform Ab^t into a natural exponential function:

$$A(1+i)^t = Ab^t = Ae^{rt}$$

For any given value of t—in this context, integer values of t—the function Ae^{rt} will, of course, yield exactly the same value as $A(1+i)^t$, such as $A(1+i) = Ae^r$ and $A(1+i)^2 = Ae^{2r}$. Consequently, even though a *discrete* case $A(1+i)^t$ is being considered, we may still work with the *continuous* natural exponential function Ae^{rt}. This explains why natural exponential functions are extensively applied in economic analysis despite the fact that not all growth patterns may actually be continuous.

Discounting and Negative Growth

Let us now turn briefly from interest compounding to the closely related concept of *discounting*. In a compound-interest problem, we seek to compute the *future value* V (principal plus interest) from a given *present value* A (initial principal).

* The method of finding the number r, given a specific value of b, will be discussed in Sec. 10.4.

The problem of *discounting* is the opposite one of finding the present value A of a given sum V which is to be available t years from now.

Let us take the discrete case first. If the amount of principal A will grow into the future value of $A(1+i)^t$ after t years of annual compounding at the interest rate i per annum, i.e., if

$$V = A(1+i)^t$$

then, by dividing both sides of the equation by the nonzero expression $(1+i)^t$, we can get the discounting formula:

$$(10.10) \qquad A = \frac{V}{(1+i)^t} = V(1+i)^{-t}$$

which involves a negative exponent. It should be realized that in this formula the roles of V and A have been reversed: V is now a given, whereas A is the unknown, to be computed from i (the rate of discount) and t (the number of years), as well as V.

Similarly, for the continuous case, if the principal A will grow into Ae^{rt} after t years of continuous compounding at the rate r in accordance with the formula

$$V = Ae^{rt}$$

then we can derive the corresponding continuous-discounting formula simply by dividing both sides of the last equation by e^{rt}:

$$(10.11) \qquad A = \frac{V}{e^{rt}} = Ve^{-rt}$$

Here again, we have A (rather than V) as the unknown, to be computed from the given future value V, the nominal rate of discount r, and the number of years t.

Taking (10.11) as an exponential growth function, we can immediately read $-r$ as the instantaneous rate of growth of A. Being negative, this rate is sometimes referred to as a *rate of decay*. Just as interest compounding exemplifies the process of growth, discounting illustrates *negative* growth.

EXERCISE 10.2

1 Use the infinite-series form of e^x in (10.6) to find the approximate value of:
(*a*) e^2 (*b*) $\sqrt{e}$ $(= e^{1/2})$
(Round off your calculation of each term to 3 decimal places, and continue with the series till you get a term 0.000.)

2 Given the function $\phi(x) = e^{2x}$:
(*a*) Write the polynomial part P_n of its Maclaurin series.
(*b*) Write the Lagrange form of the remainder R_n. Determine whether $R_n \to 0$ as $n \to \infty$, that is, whether the series is convergent to $\phi(x)$.
(*c*) If convergent, so that $\phi(x)$ may be expressed as an infinite series, write out this series.

3 Write an exponential expression for the value:
(*a*) \$10, compounded continuously at the interest rate of 5% for 3 years
(*b*) \$690, compounded continuously at the interest rate of 4% for 2 years
(These interest rates are nominal rates per annum.)

4 What is the instantaneous rate of growth of y in each of the following?
(*a*) $y = e^{0.07t}$ (*c*) $y = Ae^{0.2t}$
(*b*) $y = 12e^{0.03t}$ (*d*) $y = 0.03e^{t}$

5 Show that the two functions $y_1 = Ae^{rt}$ (interest compounding) and $y_2 = Ae^{-rt}$ (discounting) are mirror images of each other with reference to the y axis [cf. Exercise 10.1-5, part (*b*)].

10.3 LOGARITHMS

Exponential functions are closely related to *logarithmic functions* (*log functions*, for short). Before we can discuss log functions, we must first understand the meaning of the term *logarithm*.

The Meaning of Logarithm

When we have two numbers such as 4 and 16, which can be related to each other by the equation $4^2 = 16$, we define the *exponent* 2 to be the *logarithm* of 16 to the base of 4, and write

$$\log_4 16 = 2$$

It should be clear from this example that the logarithm is nothing but the *power* to which a base (4) must be raised to attain a particular number (16). In general, we may state that

$$y = b^t \quad \Leftrightarrow \quad t = \log_b y \tag{10.12}$$

which indicates that the log of y to the base b (denoted by $\log_b y$) is the power to which the base b must be raised in order to attain the value y. For this reason, it is correct, though tautological, to write

$$b^{\log_b y} = y$$

In the discussion of exponential functions, we emphasized that the function $y = b^t$ (with $b > 1$) is monotonically increasing. This means that, for any positive value of y, there is a *unique* exponent t (not necessarily positive) such that $y = b^t$; moreover, the larger the value of y, the larger must be t, as can be seen from Fig. 10.2. Translated into logarithms, the monotonicity of the exponential function implies that any positive number y must possess a *unique* logarithm t to a base $b > 1$ such that the larger the y, the larger its logarithm. As Figs. 10.1 and 10.2

show, y is necessarily positive in the exponential function $y = b^t$; consequently, a negative number or zero cannot possess a logarithm.

Common Log and Natural Log

The base of the logarithm, $b > 1$, does not have to be restricted to any particular number, but in actual log applications two numbers are widely chosen as bases—the number 10 and the number e. When 10 is the base, the logarithm is known as *common logarithm*, symbolized by $\log_{10}$ (or if the context is clear, simply by log). With e as the base, on the other hand, the logarithm is referred to as *natural logarithm* and is denoted either by $\log_e$ or by ln (for natural log). We may also use the symbol log (without subscript e) if it is not ambiguous in the particular context.

Common logarithms, used frequently in *computational* work, are exemplified by the following:

$$\log_{10} 1000 = 3 \qquad [\text{because } 10^3 = 1000]$$

$$\log_{10} 100 = 2 \qquad [\text{because } 10^2 = 100]$$

$$\log_{10} 10 = 1 \qquad [\text{because } 10^1 = 10]$$

$$\log_{10} 1 = 0 \qquad [\text{because } 10^0 = 1]$$

$$\log_{10} 0.1 = -1 \qquad [\text{because } 10^{-1} = 0.1]$$

$$\log_{10} 0.01 = -2 \qquad [\text{because } 10^{-2} = 0.01]$$

Observe the close relation between the set of numbers immediately to the left of the equals signs and the set of numbers immediately to the right. From these, it should be apparent that the common logarithm of a number between 10 and 100 must be between 1 and 2 and that the common logarithm of a number between 1 and 10 must be a positive fraction, etc. The exact logarithms can easily be obtained from a table of common logarithms or electronic calculators with log capabilities.*

In *analytical* work, however, natural logarithms prove vastly more convenient to use than common logarithms. Since, by the definition of logarithm, we have the relationship

$$(10.13) \qquad y = e^t \qquad \Leftrightarrow \qquad t = \log_e y \text{ (or } t = \ln y)$$

it is easy to see that the analytical convenience of e in exponential functions will automatically extend into the realm of logarithms with e as the base.

* More fundamentally, the value of a logarithm, like the value of e, can be calculated (or approximated) by resorting to a Maclaurin-series expansion of a log function, in a manner similar to that outlined in (10.6). However, we shall not venture into this matter here.

The following examples will serve to illustrate natural logarithms:

$$\ln e^3 = \log_e e^3 = 3$$

$$\ln e^2 = \log_e e^2 = 2$$

$$\ln e^1 = \log_e e^1 = 1$$

$$\ln 1 = \log_e e^0 = 0$$

$$\ln \frac{1}{e} = \log_e e^{-1} = -1$$

The general principle emerging from these examples is that, given an expression e^n, where n is any real number, we can automatically read the exponent n as the natural log of e^n. In general, therefore, we have the result that $\ln e^n = n$.*

Common log and natural log are convertible into each other; i.e., the base of a logarithm can be changed, just as the base of an exponential expression can. A pair of conversion formulas will be developed after we have studied the basic rules of logarithms.

Rules of Logarithms

Logarithms are in the nature of exponents; therefore, they obey certain rules closely related to the rules of exponents introduced in Sec. 2.5. These can be of great help in simplifying mathematical operations. The first three rules are stated only in terms of natural log, but they are also valid when the symbol ln is replaced by $\log_b$.

Rule I (log of a product) $\ln(uv) = \ln u + \ln v \qquad (u, v > 0)$

Example 1 $\ln(e^6e^4) = \ln e^6 + \ln e^4 = 6 + 4 = 10$

Example 2 $\ln(Ae^7) = \ln A + \ln e^7 = \ln A + 7$

PROOF By definition, $\ln u$ is the power to which e must be raised to attain the value of u; thus $e^{\ln u} = u$.† Similarly, we have $e^{\ln v} = v$ and $e^{\ln(uv)} = uv$. The latter is an exponential expression for uv. However, another expression of uv is obtainable by direct multiplication of u and v:

$$uv = e^{\ln u}e^{\ln v} = e^{\ln u + \ln v}$$

Thus, by equating the two expressions for uv, we find

$$e^{\ln(uv)} = e^{\ln u + \ln v} \qquad \text{or} \qquad \ln(uv) = \ln u + \ln v$$

* As a mnemonic device, observe that when the symbol ln (or $\log_e$) is placed at the left of the expression e^n, the symbol ln seems to cancel out the symbol e, leaving n as the answer.

† Note that when e is raised to the power $\ln u$, the symbol e and the symbol ln again seem to cancel out, leaving u as the answer.

Rule II (log of a quotient) $\ln(u/v) = \ln u - \ln v \qquad (u, v > 0)$

Example 3 $\ln(e^2/c) = \ln e^2 - \ln c = 2 - \ln c$

Example 4 $\ln(e^2/e^5) = \ln e^2 - \ln e^5 = 2 - 5 = -3$

The proof of this rule is very similar to that of Rule I and is therefore left to you as an exercise.

Rule III (log of a power) $\ln u^a = a \ln u \qquad (u > 0)$

Example 5 $\ln e^{15} = 15 \ln e = 15$

Example 6 $\ln A^3 = 3 \ln A$

PROOF By definition, $e^{\ln u} = u$; and similarly, $e^{\ln u^a} = u^a$. However, another expression for u^a can be formed as follows:

$$u^a = (e^{\ln u})^a = e^{a \ln u}$$

By equating the exponents in the two expressions for u^a, we obtain the desired result, $\ln u^a = a \ln u$.

These three rules are useful devices for simplifying the mathematical operations in certain types of problems. Rule I serves to convert, via logarithms, a multiplicative operation (uv) into an additive one ($\ln u + \ln v$); Rule II turns a division (u/v) into a subtraction ($\ln u - \ln v$); and Rule III enables us to reduce a power to a multiplicative constant. Moreover, these rules can be used in combination.

Example 7 $\ln(uv^a) = \ln u + \ln v^a = \ln u + a \ln v$

You are warned, however, that when we have *additive* expressions to begin with, logarithms may be of no help at all. In particular, it should be remembered that

$$\ln(u \pm v) \neq \ln u \pm \ln v$$

Let us now introduce two additional rules concerned with changes in the base of a logarithm.

Rule IV (conversion of log base) $\log_b u = (\log_b e)(\log_e u) \qquad (u > 0)$

This rule, which resembles the chain rule in spirit (witness the "chain" ${}_b\nearrow^e\searrow_e\nearrow^u$), enables us to derive a logarithm $\log_e u$ (to base e) from the logarithm $\log_b u$ (to base b), or vice versa.

PROOF Let $u = e^p$, so that $p = \log_e u$. Then it follows that

$$\log_b u = \log_b e^p = p \log_b e = (\log_e u)(\log_b e)$$

Rule IV can readily be generalized to

$$\log_b u = (\log_b c)(\log_c u)$$

where c is some base other than b.

Rule V (inversion of log base) $\log_b e = \dfrac{1}{\log_e b}$

This rule, which resembles the inverse-function rule of differentiation, enables us to obtain the log of b to the base e immediately upon being given the log of e to the base b, and vice versa. (This rule can also be generalized to the form $\log_b c = 1/\log_c b$).

PROOF As an application of Rule IV, let $u = b$; then we have

$$\log_b b = (\log_b e)(\log_e b)$$

But the left-side expression is $\log_b b = 1$; therefore $\log_b e$ and $\log_e b$ must be reciprocal to each other, as Rule V asserts.

From the last two rules, it is easy to derive the following pair of conversion formulas between common log and natural log:

$$\begin{aligned} \log_{10} N &= (\log_{10} e)(\log_e N) = 0.4343 \log_e N \\ \log_e N &= (\log_e 10)(\log_{10} N) = 2.3026 \log_{10} N \end{aligned} \tag{10.14}$$

for N a positive real number. The first equals sign in each formula is easily justified by Rule IV. In the first formula, the value 0.4343 (the common log of 2.71828) can be found from a table of common logarithms or an electronic calculator; in the second, the value 2.3026 (the natural log of 10) is merely the reciprocal of 0.4343, so calculated because of Rule V.

Example 8 $\log_e 100 = 2.3026(\log_{10} 100) = 2.3026(2) = 4.6052$. Conversely, we have $\log_{10} 100 = 0.4343(\log_e 100) = 0.4343(4.6052) = 2$.

An Application

The above rules of logarithms enable us to solve with ease certain simple *exponential equations* (exponential *functions* set equal to zero). For instance, if we seek to find the value of x that satisfies the equation

$$ab^x - c = 0 \qquad (a, b, c > 0)$$

we can first try to transform this exponential equation, by the use of logarithms, into a *linear* equation and then solve it as such. For this purpose, the c term should first be transposed to the right side:

$$ab^x = c$$

Whereas we do not have a simple log expression for the additive expression $(ab^x - c)$, we do have convenient log expressions for the multiplicative term ab^x and for c individually. Thus, after the transposition of c and upon taking the log (say, to base 10) of both sides, we have

$$\log a + x \log b = \log c$$

which is a linear equation in the variable x, with the solution

$$x = \frac{\log c - \log a}{\log b}$$

EXERCISE 10.3

1 What are the values of the following logarithms?
(*a*) $\log_{10} 10{,}000$ (*c*) $\log_3 81$
(*b*) $\log_{10} 0.0001$ (*d*) $\log_5 3125$

2 Evaluate the following:
(*a*) $\ln e^2$ (*c*) $\ln(1/e^3)$ (*e*) $(e^{\ln 3})!$
(*b*) $\log_e e^{-4}$ (*d*) $\log_e(1/e^2)$ (*f*) $\ln e^x - e^{\ln x}$

3 Evaluate the following by application of the rules of logarithms:
(*a*) $\log_{10}(100)^{14}$ (*d*) $\ln Ae^2$
(*b*) $\log_{10} \frac{1}{100}$ (*e*) $\ln ABe^{-4}$
(*c*) $\ln(3/B)$ (*f*) $(\log_4 e)(\log_e 64)$

4 Which of the following are valid?
(*a*) $\ln u - 2 = \ln \dfrac{u}{e^2}$ (c) $\ln u + \ln v - \ln w = \ln \dfrac{uv}{w}$
(*b*) $3 + \ln v = \ln \dfrac{e^3}{v}$ (d) $\ln 3 + \ln 5 = \ln 8$

5 Prove that $\ln(u/v) = \ln u - \ln v$.

10.4 LOGARITHMIC FUNCTIONS

When a variable is expressed as a function of the logarithm of another variable, the function is referred to as a *logarithmic function*. We have already seen two versions of this type of function in (10.12) and (10.13), namely,

$$t = \log_b y \qquad \text{and} \qquad t = \log_e y \; (= \ln y)$$

which differ from each other only in regard to the base of the logarithm.

Log Functions and Exponential Functions

As we stated earlier, log functions are inverse functions of certain exponential functions. An examination of the above two log functions will confirm that they are indeed the respective inverse functions of the exponential functions

$$y = b^t \qquad \text{and} \qquad y = e^t$$

because the log functions cited are the results of reversing the roles of the dependent and independent variables of the corresponding exponential functions. You should realize, of course, that the symbol t is being used here as a general symbol, and it does not necessarily stand for *time*. Even when it does, its appearance as a *dependent* variable does not mean that time is determined by some variable y; it means only that a given value of y is associated with a unique point of time.

As inverse functions of monotonically increasing (exponential) functions, logarithmic functions must also be monotonically increasing, which is consistent with our earlier statement that the larger a number, the larger is its logarithm to any given base. This property may be expressed symbolically in terms of the following two propositions: For two positive values of y (y_1 and y_2),

$$\text{(10.15)} \qquad \begin{aligned} \ln y_1 = \ln y_2 \quad &\Leftrightarrow \quad y_1 = y_2 \\ \ln y_1 > \ln y_2 \quad &\Leftrightarrow \quad y_1 > y_2 \end{aligned}$$

These propositions are also valid, of course, if we replace ln by $\log_b$.

The Graphical Form

The monotonicity and other general properties of logarithmic functions can be clearly observed from their graphs. Given the graph of the exponential function $y = e^t$, we can obtain the graph of the corresponding log function by replotting the original graph with the two axes transposed. The result of such replotting is illustrated in Fig. 10.3. Note that if diagram b were laid over diagram a, with y axis on y axis and t axis on t axis, the two curves should coincide exactly. As they actually appear in Fig. 10.3—with interchanged axes—on the other hand, the two curves are seen to be mirror images of each other (as the graphs of any pair of inverse functions must be) with reference to the 45° line drawn through the origin.

This mirror-image relationship has several noteworthy implications. For one, although both are monotonically increasing, the log curve increases at a *decreasing rate* (second derivative negative), in contradistinction to the exponential curve, which increases at an increasing rate. Another interesting contrast is that, while the exponential function has a positive *range*, the log function has a positive *domain* instead. (This latter restriction on the domain of the log function is, of course, merely another way of stating that only positive numbers possess logarithms.) A third consequence of the mirror-image relationship is that, just as $y = e^t$ has a vertical intercept at 1, the log function $t = \log_e y$ must cross the

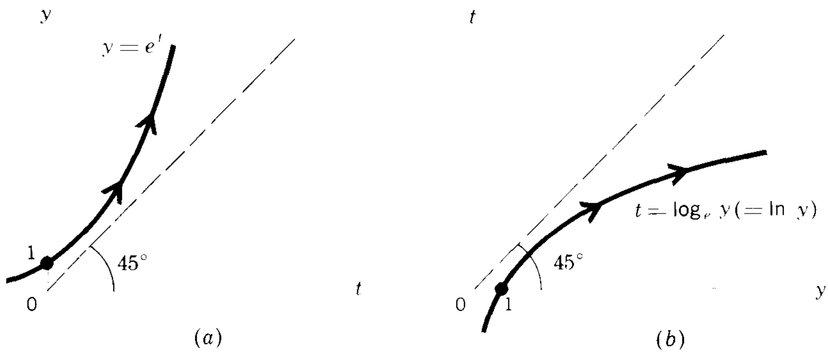

Figure 10.3

horizontal axis at $y = 1$, indicating that $\log_e 1 = 0$. Inasmuch as this horizontal intercept is unaffected by the base of the logarithm—for instance, $\log_{10} 1 = 0$ also—we may infer from the general shape of the log curve in Fig. 10.3*b* that, for *any* base,

$$(10.16) \qquad \left.\begin{matrix} 0 < y < 1 \\ y = 1 \\ y > 1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \log y < 0 \\ \log y = 0 \\ \log y > 0 \end{matrix}\right.$$

For verification, we can check the two sets of examples of common and natural logarithms given in Sec. 10.3. Furthermore, we may note that

$$(10.16') \qquad \log y \to \left\{\begin{matrix} \infty \\ -\infty \end{matrix}\right\} \qquad \text{as } y \to \left\{\begin{matrix} \infty \\ 0^+ \end{matrix}\right.$$

The graphical comparison of the logarithmic function and the exponential function in Fig. 10.3 is based on the simple functions $y = e^t$ and $t = \ln y$. The same general result will prevail if we compare the generalized exponential function $y = Ae^{rt}$ with its corresponding log function. With the (positive) constants A and r to *compress* or *extend* the exponential curve, it will nevertheless resemble the general shape of Fig. 10.3*a*, except that its vertical intercept will be at $y = A$ rather than at $y = 1$ (when $t = 0$, we have $y = Ae^0 = A$). Its inverse function, accordingly, must have a *horizontal* intercept at $y = A$. In general, with reference to the 45° line, the corresponding log curve will be a mirror image of the exponential curve.

If the specific algebraic expression of the inverse of $y = Ae^{rt}$ is desired, it can be obtained by taking the natural log of both sides of this exponential function [which, according to the first proposition in (10.15), will leave the equation undisturbed] and then solving for t:

$$\ln y = \ln(Ae^{rt}) = \ln A + rt \ln e = \ln A + rt$$

hence

$$(10.17) \qquad t = \frac{\ln y - \ln A}{r} \qquad (r \neq 0)$$

This result, a log function, constitutes the inverse of the exponential function $y = Ae^{rt}$. As claimed earlier, the function in (10.17) has a horizontal intercept at $y = A$, because when $y = A$, we have $\ln y = \ln A$, and therefore $t = 0$.

Base Conversion

In Sec. 10.2, it was stated that the exponential function $y = Ab^t$ can always be converted into a *natural* exponential function $y = Ae^{rt}$. We are now ready to derive a conversion formula. Instead of Ab^t, however, let us consider the conversion of the more general expression Ab^{ct} into Ae^{rt}. Since the essence of the problem is to find an r from given values of b and c such that

$$e^r = b^c$$

all that is necessary is to express r as a function of b and c. Such a task is easily accomplished by taking the natural log of both sides of the last equation:

$$\ln e^r = \ln b^c$$

The left side can immediately be read as equal to r, so that the desired function (conversion formula) emerges as

$$r = \ln b^c = c \ln b \qquad (10.18)$$

This indicates that the function $y = Ab^{ct}$ can always be rewritten in the natural-base form, $y = Ae^{(c \ln b)t}$

Example 1 Convert $y = 2^t$ to a natural exponential function. Here, we have $A = 1$, $b = 2$, and $c = 1$. Hence $r = c \ln b = \ln 2$, and the desired exponential function is

$$y = Ae^{rt} = e^{(\ln 2)t}$$

If we like, we can also calculate the numerical value of (ln 2) by use of (10.14) and a table of common logarithms as follows:

$$\ln 2 = 2.3026 \log_{10} 2 = 2.3026(0.3010) = 0.6931 \qquad (10.19)$$

Then we may express the earlier result alternatively as $y = e^{0.6931t}$.

Example 2 Convert $y = 3(5)^{2t}$ to a natural exponential function. In this example, $A = 3$, $b = 5$, and $c = 2$, and formula (10.18) gives us $r = 2 \ln 5$. Therefore the desired function is

$$y = Ae^{rt} = 3e^{(2 \ln 5)t}$$

Again, if we like, we can calculate that

$$2 \ln 5 = \ln 25 = 2.3026 \log_{10} 25 = 2.3026(1.3979) = 3.2188$$

so the earlier result can be alternatively expressed as $y = 3e^{3.2188t}$

It is also possible, of course, to convert log functions of the form $t = \log_b y$ into equivalent natural log functions. To that end, it is sufficient to apply Rule IV of logarithms, which may be expressed as

$$\log_b y = (\log_b e)(\log_e y)$$

The direct substitution of this result into the given log function will immediately give us the desired natural log function:

$$t = \log_b y = (\log_b e)(\log_e y)$$
$$= \frac{1}{\log_e b}\log_e y \qquad \text{[by Rule V of logarithms]}$$
$$= \frac{\ln y}{\ln b}$$

By the same procedure, we can transform the more general log function $t = a\log_b(cy)$ into the equivalent form

$$t = a(\log_b e)(\log_e cy) = \frac{a}{\log_e b}\log_e(cy) = \frac{a}{\ln b}\ln(cy)$$

Example 3 Convert the function $t = \log_2 y$ into the natural log form. Since in this example we have $b = 2$ and $a = c = 1$, the desired function is

$$t = \frac{1}{\ln 2}\ln y$$

By (10.19), however, we may also express it as $t = (1/0.6931)\ln y$.

Example 4 Convert the function $t = 7\log_{10} 2y$ into a natural logarithmic function. The values of the constants are in this case $a = 7$, $b = 10$, and $c = 2$; consequently, the desired function is

$$t = \frac{7}{\ln 10}\ln 2y$$

But since $\ln 10 = 2.3026$, as (10.14) indicates, the above function can be rewritten as $t = (7/2.3026)\ln 2y = 3.0400\ln 2y$.

In the above discussion, we have followed the practice of expressing t as a function of y when the function is logarithmic. The only reason for doing so is our desire to stress the inverse-function relationship between the exponential and logarithmic functions. When a log function is studied *by itself*, we shall write $y = \ln t$ (rather than $t = \ln y$), as is customary. Naturally, nothing in the analytical aspect of the discussion will be affected by such an interchange of symbols.

EXERCISE 10.4

1 The form of the inverse function of $y = Ae^{rt}$ in (10.17) requires r to be nonzero. What is the meaning of this requirement when viewed in reference to the original exponential function $y = Ae^{rt}$?

2 (*a*) Sketch a graph of the exponential function $y = Ae^{rt}$; indicate the value of the vertical intercept.

(*b*) Then sketch the graph of the log function $t = \dfrac{\ln y - \ln A}{r}$, and indicate the value of the horizontal intercept.

3 Find the inverse function of $y = ab^{ct}$.

4 Transform the following functions to their natural exponential forms:

(*a*) $y = 8^{3t}$ (*c*) $y = 5(5)^t$

(*b*) $y = 2(7)^{2t}$ (*d*) $y = 2(15)^{4t}$

5 Transform the following functions to their natural logarithmic forms:

(*a*) $t = \log_7 y$ (*c*) $t = 3\log_{15} 9y$

(*b*) $t = \log_8 3y$ (*d*) $t = 2\log_{10} y$

6 Find the continuous-compounding nominal interest rate per annum (r) that is equivalent to a discrete-compounding interest rate (i) of

(*a*) 5 percent per annum, compounded annually

(*b*) 5 percent per annum, compounded semiannually

(*c*) 6 percent per annum, compounded semiannually

(*d*) 6 percent per annum, compounded quarterly

10.5 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Earlier it was claimed that the function e^t is its own derivative. As it turns out, the natural log function, $\ln t$, possesses a rather convenient derivative also, namely, $d(\ln t)/dt = 1/t$. This fact reinforces our preference for the base e. Let us now prove the validity of these two derivative formulas, and then we shall deduce the derivative formulas for certain variants of the exponential and log expressions e^t and $\ln t$.

Log-Function Rule

The derivative of the log function $y = \ln t$ is

$$\frac{d}{dt}\ln t = \frac{1}{t}$$

To prove this, we recall that, by definition, the derivative of $y = f(t) = \ln t$ has the following value at $t = N$:

$$f'(N) = \lim_{t \to N} \frac{f(t) - f(N)}{t - N} = \lim_{t \to N} \frac{\ln t - \ln N}{t - N} = \lim_{t \to N} \frac{\ln(t/N)}{t - N}$$

[by Rule II of logarithms]

Now let us introduce a shorthand symbol $m \equiv \dfrac{N}{t - N}$. Then we can write $\dfrac{1}{t - N} =$

$\frac{m}{N}$, and also $\frac{t}{N} = 1 + \frac{t-N}{N} = 1 + \frac{1}{m}$. Thus the expression to the right of the limit sign above can be converted to the form

$$\frac{1}{t-N}\ln\frac{t}{N} = \frac{m}{N}\ln\left(1 + \frac{1}{m}\right) = \frac{1}{N}\ln\left(1 + \frac{1}{m}\right)^m$$

[by Rule III of logarithms]

Note that, when t tends to N, m will tend to infinity. Thus, to find the desired derivative value, we may take the limit of the last expression above as $m \to \infty$:

$$f'(N) = \lim_{m\to\infty}\frac{1}{N}\ln\left(1 + \frac{1}{m}\right)^m = \frac{1}{N}\ln e = \frac{1}{N} \qquad \text{[by (10.5)]}$$

Since N can be any number for which a logarithm is defined, however, we can generalize this result, and write $f'(t) = d(\ln t)/dt = 1/t$. This proves the log-function rule.

Exponential-Function Rule

The derivative of the function $y = e^t$ is

$$\frac{d}{dt}e^t = e^t$$

This result follows easily from the log-function rule. We know that the inverse function of the function $y = e^t$ is $t = \ln y$, with derivative $dt/dy = 1/y$. Thus, by the inverse-function rule, we may write immediately

$$\frac{d}{dt}e^t = \frac{dy}{dt} = \frac{1}{dt/dy} = \frac{1}{1/y} = y = e^t$$

The Rules Generalized

The above two rules can be generalized to cases where the variable t in the expression e^t and $\ln t$ is replaced by some *function* of t, say, $f(t)$. The generalized versions of the two rules are

$$\text{(10.20)} \qquad \begin{aligned} &\frac{d}{dt}e^{f(t)} = f'(t)e^{f(t)} &&\left[\text{or } \frac{d}{dt}e^u = e^u\frac{du}{dt}\right] \\ &\frac{d}{dt}\ln f(t) = \frac{f'(t)}{f(t)} &&\left[\text{or } \frac{d}{dt}\ln v = \frac{1}{v}\frac{dv}{dt}\right] \end{aligned}$$

The proofs for (10.20) involve nothing more than the straightforward application of the chain rule. Given a function $y = e^{f(t)}$, we can first let $u = f(t)$, so that $y = e^u$. Then, by the chain rule, the derivative emerges as

$$\frac{d}{dt}e^{f(t)} = \frac{d}{dt}e^u = \frac{d}{du}e^u\frac{du}{dt} = e^u\frac{du}{dt} = e^{f(t)}f'(t)$$

Similarly, given a function $y = \ln f(t)$, we can first let $v = f(t)$, so as to form a chain: $y = \ln v$, where $v = f(t)$. Then, by the chain rule, we have

$$\frac{d}{dt}\ln f(t) = \frac{d}{dt}\ln v = \frac{d}{dv}\ln v\frac{dv}{dt} = \frac{1}{v}\frac{dv}{dt} = \frac{1}{f(t)}f'(t)$$

Note that the only real modification introduced in (10.20) beyond the simpler rules $de^t/dt = e^t$ and $d(\ln t)/dt = 1/t$ is the multiplicative factor $f'(t)$.

Example 1 Find the derivative of the function $y = e^{rt}$. Here, the exponent is $rt = f(t)$, with $f'(t) = r$; thus

$$\frac{dy}{dt} = \frac{d}{dt}e^{rt} = re^{rt}$$

Example 2 Find dy/dt from the function $y = e^{-t}$. In this case, $f(t) = -t$, so that $f'(t) = -1$. As a result,

$$\frac{dy}{dt} = \frac{d}{dt}e^{-t} = -e^{-t}$$

Example 3 Find dy/dt from the function $y = \ln at$. Since in this case $f(t) = at$, with $f'(t) = a$, the derivative is

$$\frac{d}{dt}\ln at = \frac{a}{at} = \frac{1}{t}$$

which is, interestingly enough, identical with the derivative of $y = \ln t$.

This example illustrates the fact that a multiplicative constant for t *within* a log expression drops out in the process of derivation. But note that, for a constant k, we have

$$\frac{d}{dt}k\ln t = k\frac{d}{dt}\ln t = \frac{k}{t}$$

thus a multiplicative constant *without* the log expression is still retained in derivation.

Example 4 Find the derivative of the function $y = \ln t^c$. With $f(t) = t^c$ and $f'(t) = ct^{c-1}$, the formula in (10.20) yields

$$\frac{d}{dt}\ln t^c = \frac{ct^{c-1}}{t^c} = \frac{c}{t}$$

Example 5 Find dy/dt from $y = t^3\ln t^2$. Because this function is a product of two terms t^3 and $\ln t^2$, the product rule should be used:

$$\begin{aligned}\frac{dy}{dt} &= t^3\frac{d}{dt}\ln t^2 + \ln t^2\frac{d}{dt}t^3\\ &= t^3\left(\frac{2t}{t^2}\right) + (\ln t^2)(3t^2)\\ &= 2t^2 + 3t^2(2\ln t) \qquad \text{[Rule III of logarithms]}\\ &= 2t^2(1 + 3\ln t)\end{aligned}$$

The Case of Base b

For exponential and log functions with base b, the derivatives are

$$\frac{d}{dt}b^t = b^t \ln b \qquad \left[\textit{Warning}: \frac{d}{dt}b^t \neq tb^{t-1}\right]$$

(10.21)

$$\frac{d}{dt}\log_b t = \frac{1}{t\ln b}$$

Note that in the special case of base e (when $b = e$), we have $\ln b = \ln e = 1$, so that these two derivatives will reduce to $(d/dt)e^t = e^t$ and $(d/dt)\ln t = 1/t$, respectively.

The proofs for (10.21) are not difficult. For the case of b^t, the proof is based on the identity $b \equiv e^{\ln b}$, which enables us to write

$$b^t = e^{(\ln b)t} = e^{t\ln b}$$

(We write $t\ln b$, instead of $\ln b\, t$, in order to emphasize that t is not a part of the log expression.) Hence

$$\frac{d}{dt}b^t = \frac{d}{dt}e^{t\ln b} = (\ln b)(e^{t\ln b}) \qquad [\text{by } (10.20)]$$

$$= (\ln b)(b^t) = b^t \ln b$$

To prove the second part of (10.21), on the other hand, we rely on the basic log property that

$$\log_b t = (\log_b e)(\log_e t) = \frac{1}{\ln b}\ln t$$

which leads us to the derivative

$$\frac{d}{dt}\log_b t = \frac{d}{dt}\left(\frac{1}{\ln b}\ln t\right) = \frac{1}{\ln b}\frac{d}{dt}\ln t = \frac{1}{\ln b}\left(\frac{1}{t}\right)$$

The more general versions of these two formulas are

$$\frac{d}{dt}b^{f(t)} = f'(t)b^{f(t)}\ln b$$

(10.21′)

$$\frac{d}{dt}\log_b f(t) = \frac{f'(t)}{f(t)}\frac{1}{\ln b}$$

Again, it is seen that if $b = e$, then $\ln b = 1$, and these formulas will reduce to (10.20).

Example 6 Find the derivative of the function $y = 12^{1-t}$. Here, $b = 12$, $f(t) = 1 - t$, and $f'(t) = -1$; thus

$$\frac{dy}{dt} = -(12)^{1-t}\ln 12$$

Higher Derivatives

Higher derivatives of exponential and log functions, like those of other types of functions, are merely the results of repeated differentiation.

Example 7 Find the *second* derivative of $y = b^t$ (with $b > 1$). The first derivative, by (10.21), is $y'(t) = b^t \ln b$ (where $\ln b$ is, of course, a constant); thus, by differentiating once more with respect to t, we have

$$y''(t) = \frac{d}{dt}y'(t) = \left(\frac{d}{dt}b^t\right)\ln b = (b^t \ln b)\ln b = b^t(\ln b)^2$$

Note that $y = b^t$ is always positive and $\ln b$ (for $b > 1$) is also positive [by (10.16)]; thus $y'(t) = b^t \ln b$ must be positive. And $y''(t)$, being a product of b^t and a squared number, is also positive. These facts confirm our previous statement that the exponential function $y = b^t$ increases monotonically at an increasing rate.

Example 8 Find the *second* derivative of $y = \ln t$. The first derivative is $y' = 1/t = t^{-1}$; hence, the second derivative is

$$y'' = -t^{-2} = \frac{-1}{t^2}$$

Inasmuch as the domain of this function consists of the open interval $(0, \infty)$, $y' = 1/t$ must be a positive number. On the other hand, y'' is always negative. Together, these conclusions serve to confirm our earlier allegation that the log function $y = \ln t$ increases monotonically at a decreasing rate.

An Application

One of the prime virtues of the logarithm is its ability to convert a multiplication into an addition, and a division into a subtraction. This property can be exploited when we are differentiating a complicated product or quotient of any type of functions (not necessarily exponential or logarithmic).

Example 9 Find dy/dx from

$$y = \frac{x^2}{(x+3)(2x+1)}$$

Instead of applying the product and quotient rules, we may first take the natural log of both sides of the equation to reduce the function to the form

$$\ln y = \ln x^2 - \ln(x+3) - \ln(2x+1)$$

According to (10.20), the derivative of the left side with respect to x is

$$\frac{d}{dx}\text{(left side)} = \frac{1}{y}\frac{dy}{dx}$$

whereas the right side gives

$$\frac{d}{dx}\text{(right side)} = \frac{2x}{x^2} - \frac{1}{x+3} - \frac{2}{2x+1} = \frac{7x+6}{x(x+3)(2x+1)}$$

When the two results are equated and both sides are multiplied by y, we get the desired derivative as follows:

$$\frac{dy}{dx} = \frac{7x+6}{x(x+3)(2x+1)}y$$

$$= \frac{7x+6}{x(x+3)(2x+1)}\,\frac{x^2}{(x+3)(2x+1)} = \frac{x(7x+6)}{(x+3)^2(2x+1)^2}$$

Example 10 Find dy/dx from $y = x^a e^{kx-c}$. Taking the natural log of both sides, we have

$$\ln y = a\ln x + \ln e^{kx-c} = a\ln x + kx - c$$

Differentiating both sides with respect to x, and using (10.20), we then get

$$\frac{1}{y}\frac{dy}{dx} = \frac{a}{x} + k$$

and $$\frac{dy}{dx} = \left(\frac{a}{x} + k\right)y = \left(\frac{a}{x} + k\right)x^a e^{kx-c}$$

EXERCISE 10.5

1 Find the derivatives of:

(a) $y = e^{2t+4}$ $\quad$ (e) $y = e^{ax^2+bx+c}$

(b) $y = e^{1-7t}$ $\quad$ (f) $y = xe^x$

(c) $y = e^{t^2+1}$ $\quad$ (g) $y = x^2e^{2x}$

(d) $y = 3e^{2-t^2}$ $\quad$ (h) $y = axe^{bx+c}$

2 (a) Verify the derivative in Example 3 by utilizing the equation $\ln at = \ln a + \ln t$.
(b) Verify the result in Example 4 by utilizing the equation $\ln t^c = c\ln t$.

3 Find the derivatives of:

(a) $y = \ln 8t^5$ $\quad$ (e) $y = \ln x - \ln(1+x)$

(b) $y = \ln at^c$ $\quad$ (f) $y = \ln[x(1-x)^8]$

(c) $y = \ln(t+9)$ $\quad$ (g) $y = \ln\left(\frac{3x}{1+x}\right)$

(d) $y = 5\ln(t+1)^2$ $\quad$ (h) $y = 5x^4\ln x^2$

4 Find the derivatives of:

(a) $y = 5^t$ $\quad$ (d) $y = \log_7 7x^2$

(b) $y = \log_2(t+1)$ $\quad$ (e) $y = \log_2(8x^2+3)$

(c) $y = 13^{2t+3}$ $\quad$ (f) $y = x^2\log_3 x$

5 Prove the two formulas in (10.21′).

6 Show that the function $V = Ae^{rt}$ (with $A, r > 0$) and the function $A = Ve^{-rt}$ (with $V, r > 0$) are both monotonic, but in opposite directions, and that they are both strictly convex in shape (cf. Exercise 10.2-5).

7 Find the derivatives of the following by first taking the natural log of both sides:

(*a*) $y = \dfrac{3x}{(x+2)(x+4)}$ (*b*) $y = (x^2+3)e^{x^2+1}$

10.6 OPTIMAL TIMING

What we have learned about exponential and log functions can now be applied to some simple problems of optimal timing.

A Problem of Wine Storage

Suppose that a certain wine dealer is in possession of a particular quantity (say, a case) of wine, which he can either sell at the present time ($t = 0$) for a sum of $\$K$ or else store for a variable length of time and then sell at a higher value. The growing value (V) of the wine is known to be the following function of time:

$$(10.22) \qquad V = Ke^{\sqrt{t}} \qquad \left[= K\exp(t^{1/2})\right]$$

so that if $t = 0$ (sell now), then $V = K$. The problem is to ascertain when he should sell it in order to maximize profit, assuming the storage cost to be nil.*

Since the cost of wine is a "sunk" cost—the wine is already paid for by the dealer—and since storage cost is assumed to be nonexistent, to maximize profit is the same as maximizing the sales revenue, or the value of V. There is one catch, however. Each value of V corresponding to a specific point of t represents a dollar sum receivable at a different date and, because of the interest element involved, is not directly comparable with the V value of another date. The way out of this difficulty is to *discount* each V figure to its *present-value* equivalent (the value at time $t = 0$), for then all the V values will be on a comparable footing.

Let us assume that the interest rate on the continuous-compounding basis is at the level of r. Then, according to (10.11), the present value of V can be expressed as

$$(10.22') \qquad A(t) = Ve^{-rt} = Ke^{\sqrt{t}}e^{-rt} = Ke^{\sqrt{t}-rt}$$

where A, denoting the present value of V, is itself a function of t. Therefore our problem amounts to finding the value of t that maximizes A.

* The consideration of storage cost will entail a difficulty we are not yet equipped to handle. Later, in Chap. 13, we shall return to this problem.