CHAPTER

TWELVE

OPTIMIZATION WITH EQUALITY CONSTRAINTS

The last chapter presented a general method for finding the relative extrema of an objective function of two or more choice variables. One important feature of that discussion is that all the choice variables are *independent* of one another, in the sense that the decision made regarding one variable does not impinge upon the choices of the remaining variables. For instance, a two-product firm can choose any value for Q_1 and any value for Q_2 it wishes, without the two choices limiting each other.

If the said firm is somehow required to observe a restriction (such as a production quota) in the form of $Q_1 + Q_2 = 950$, however, the independence between the choice variables will be lost. In that event, the firm's profit-maximizing output levels $\bar{Q}_1$ and $\bar{Q}_2$ will be not only simultaneous but also dependent, because the higher $\bar{Q}_1$ is, the lower $\bar{Q}_2$ must correspondingly be, in order to stay within the combined quota of 950. The new optimum satisfying the production quota constitutes a *constrained optimum*, which, in general, may be expected to differ from the *free optimum* discussed in the preceding chapter.

A restriction, such as the production quota mentioned above, establishes a relationship between the two variables in their roles as choice variables, but this should be distinguished from other types of relationships that may link the variables together. For instance, in Example 2 of Sec. 11.6, the two products of the firm are related in consumption (substitutes) as well as in production (as is reflected in the cost function), but that fact does not qualify the problem as one of constrained optimization, since the two output variables are still *independent as* *choice variables*. Only the dependence of the variables qua choice variables gives rise to a constrained optimum.

In the present chapter, we shall consider equality constraints only, such as $Q_1 + Q_2 = 950$. Our primary concern will be with *relative* constrained extrema, although *absolute* ones will also be discussed in Sec. 12.4.

12.1 EFFECTS OF A CONSTRAINT

The primary purpose of imposing a constraint is to give due cognizance to certain limiting factors present in the optimization problem under discussion.

We have already seen the limitation on output choices that result from a production quota. For further illustration, let us consider a consumer with the simple utility (index) function

$$U = x_1 x_2 + 2x_1 \tag{12.1}$$

Since the marginal utilities—the partial derivatives $U_1 \equiv \partial U/\partial x_1$ and $U_2 \equiv \partial U/\partial x_2$—are positive for all positive levels of x_1 and x_2 here, to have U maximized without any constraint, the consumer should purchase an *infinite* amount of both goods, a solution that obviously has little practical relevance. To render the optimization problem meaningful, the purchasing power of the consumer must also be taken into account; i.e., a *budget constraint* should be incorporated into the problem. If the consumer intends to spend a given sum, say, \$60, on the two goods and if the current prices are $P_{10} = 4$ and $P_{20} = 2$, then the budget constraint can be expressed by the linear equation

$$4x_1 + 2x_2 = 60 \tag{12.2}$$

Such a constraint, like the production quota referred to earlier, renders the choices of $\bar{x}_1$ and $\bar{x}_2$ mutually dependent.

The problem now is to maximize (12.1), subject to the constraint stated in (12.2). Mathematically, what the constraint (variously called *restraint*, *side relation*, or *subsidiary condition*) does is to narrow the domain, and hence the range of the objective function. The domain of (12.1) would normally be the set $\{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}$. Graphically, the domain is represented by the nonnegative quadrant of the x_1x_2 plane in Fig. 12.1*a*. After the budget constraint (12.2) is added, however, we can admit only those values of the variables which satisfy this latter equation, so that the domain is immediately reduced to the set of points lying on the budget line. This will automatically affect the range of the objective function, too; only that subset of the utility surface lying directly above the budget-constraint line will now be relevant. The said subset (a cross section of the surface) may look like the curve in Fig. 12.1*b*, where U is plotted on the vertical axis, with the budget line of diagram *a* placed on the horizontal axis. Our interest, then, is only in locating the maximum on the curve in diagram *b*.

In general, for a function $z = f(x, y)$, the difference between a constrained extremum and a free extremum may be illustrated in the three-dimensional graph

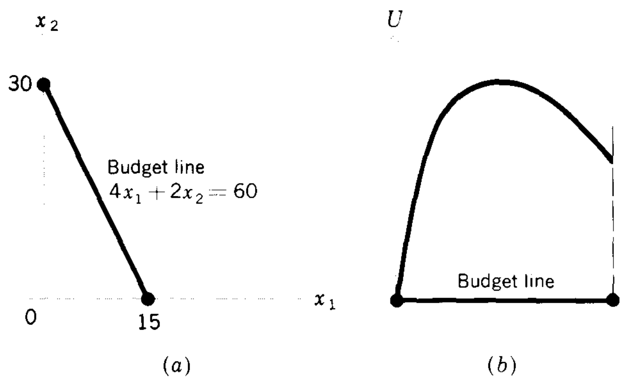

Figure 12.1

of Fig. 12.2. The free extremum in this particular graph is the peak point of the entire dome, but the constrained extremum is at the peak of the inverse U-shaped curve situated on top of (i.e., lying directly above) the constraint line. In general, a constrained maximum can be expected to have a lower value than the free maximum, although, by coincidence, the two maxima may happen to have the same value. But the constrained maximum can never exceed the free maximum.

It is interesting to note that, had we added another constraint intersecting the first constraint at a single point in the xy plane, the two constraints together would have restricted the domain to that single point. Then the locating of the

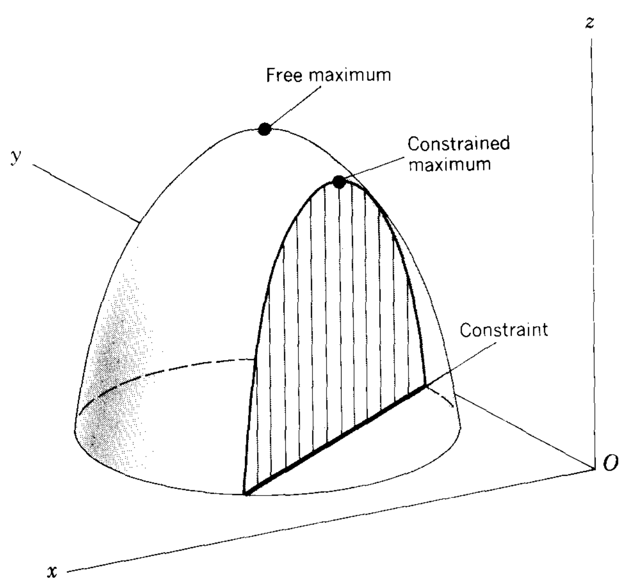

Figure 12.2

extremum would become a trivial matter. In a meaningful problem, the number and the nature of the constraints should be such as to restrict, but not eliminate, the possibility of choice. Generally, the number of constraints should be less than the number of choice variables.

12.2 FINDING THE STATIONARY VALUES

Even without any new technique of solution, the constrained maximum in the simple example defined by (12.1) and (12.2) can easily be found. Since the constraint (12.2) implies

$$x_2 = \frac{60 - 4x_1}{2} = 30 - 2x_1 \qquad (12.2')$$

we can combine the constraint with the objective function by substituting (12.2′) into (12.1). The result is an objective function in one variable only:

$$U = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2x_1^2$$

which can be handled with the method already learned. By setting $dU/dx_1 = 32 - 4x_1$ equal to zero, we get the solution $\bar{x}_1 = 8$, which by virtue of (12.2′) immediately leads to $\bar{x}_2 = 30 - 2(8) = 14$. From (12.1), we can then find the stationary value $\bar{U} = 128$; and since the second derivative is $d^2U/dx_1^2 = -4 < 0$, that stationary value constitutes a (constrained) maximum of U.*

When the constraint is itself a complicated function, or when there are several constraints to consider, however, the technique of substitution and elimination of variables could become a burdensome task. More importantly, when the constraint comes in a form such that we cannot solve it to express one variable (x_2) as an explicit function of the other (x_1), the elimination method would in fact be of no avail—even if x_2 were known to be an implicit function of x_1, that is, even if the conditions of the implicit-function theorem were satisfied. In such cases, we may resort to a method known as the *method of Lagrange (undetermined) multiplier*, which, as we shall see, has distinct analytical advantages.

Lagrange-Multiplier Method

The essence of the Lagrange-multiplier method is to convert a constrained-extremum problem into a form such that the first-order condition of the free-extremum problem can still be applied.

Given the problem of maximizing $U = x_1x_2 + 2x_1$, subject to the constraint $4x_1 + 2x_2 = 60$ [from (12.1) and (12.2)], let us write what is referred to as the *Lagrangian function*, which is a modified version of the objective function that

* You may recall that for the flower-bed problem of Exercise 9.4-2 the same technique of substitution was applied to find the maximum area, using a constraint (the available quantity of wire netting) to eliminate one of the two variables (the length or the width of the flower bed).

incorporates the constraint as follows:

$$(12.3) \qquad Z = x_1 x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$$

The symbol λ (the Greek letter lambda), representing some as yet undetermined number, is called a *Lagrange* (*undetermined*) *multiplier*. If we can somehow be assured that $4x_1 + 2x_2 = 60$, so that the constraint will be satisfied, then the last term in (12.3) will vanish regardless of the value of λ. In that event, Z will be identical with U. Moreover, with the constraint out of the way, we only have to seek the *free* maximum of Z, in lieu of the *constrained* maximum of U, with respect to the two variables x_1 and x_2. The question is: How can we make the parenthetical expression in (12.3) vanish?

The tactic that will accomplish this is simply to treat λ as an additional variable in (12.3), i.e., to consider $Z = Z(\lambda, x_1, x_2)$. For then the first-order condition for free extremum will consist of the set of simultaneous equations

$$(12.4) \qquad \begin{aligned} Z_\lambda(\equiv \partial Z/\partial\lambda\ ) &= 60 - 4x_1 - 2x_2 = 0 \\ Z_1(\equiv \partial Z/\partial x_1) &= x_2 + 2 - 4\lambda = 0 \\ Z_2(\equiv \partial Z/\partial x_2) &= x_1 - 2\lambda = 0 \end{aligned}$$

and the first equation will automatically guarantee the satisfaction of the constraint. Thus, by incorporating the constraint into the Lagrangian function Z and by treating the Lagrange multiplier as an extra variable, we can obtain the constrained extremum $\bar{U}$ (two choice variables) simply by screening the stationary values of Z, taken as a *free* function of three choice variables.

Solving (12.4) for the critical values of the variables, we find $\bar{x}_1 = 8$, $\bar{x}_2 = 14$ (and $\bar{\lambda} = 4$). As expected, the values of $\bar{x}_1$ and $\bar{x}_2$ check with the answers already obtained by the substitution method. Furthermore, it is clear from (12.3) that $\bar{Z} = 128$; this is identical with the value of $\bar{U}$ found earlier, as it should be.

In general, given an objective function

$$(12.5) \qquad z = f(x, y)$$

subject to the constraint

$$(12.6) \qquad g(x, y) = c$$

where c is a constant,* we can write the Lagrangian function as

$$(12.7) \qquad Z = f(x, y) + \lambda[c - g(x, y)]$$

For stationary values of Z, regarded as a function of the three variables λ, x, and

* It is also possible to subsume the constant c under the contraint function so that (12.6) appears instead as $G(x, y) = 0$, where $G(x, y) = g(x, y) - c$. In that case, (12.7) should be changed to $Z = f(x, y) + \lambda[0 - G(x, y)] = f(x, y) - \lambda G(x, y)$. The version in (12.6) is chosen because it facilitates the study of the comparative-static effect of a change in the constraint constant later [see (12.16)].

y, the necessary condition is

$$\begin{aligned} & Z_\lambda = c - g(x, y) = 0 \\ (12.8) \qquad & Z_x = f_x - \lambda g_x = 0 \\ & Z_y = f_y - \lambda g_y = 0 \end{aligned}$$

Since the first equation in (12.8) is simply a restatement of (12.6), the stationary values of the Lagrangian function Z will automatically satisfy the constraint of the original function z. And since the expression $\lambda[c - g(x, y)]$ is now assuredly zero, the stationary values of Z in (12.7) must be identical with those of (12.5), subject to (12.6).

Let us illustrate the method with two more examples.

Example 1 Find the extremum of

$$z = xy \qquad \text{subject to} \qquad x + y = 6$$

The first step is to write the Lagrangian function

$$Z = xy + \lambda(6 - x - y)$$

For a stationary value of Z, it is necessary that

$$\left.\begin{aligned} Z_\lambda &= 6 - x - y = 0 \\ Z_x &= y - \lambda = 0 \\ Z_y &= x - \lambda = 0 \end{aligned}\right\} \qquad \text{or} \qquad \left\{\begin{aligned} x + y &= 6 \\ -\lambda \qquad + y &= 0 \\ -\lambda + x \qquad &= 0 \end{aligned}\right.$$

Thus, by Cramer's rule or some other method, we can find

$$\bar{\lambda} = 3 \qquad \bar{x} = 3 \qquad \bar{y} = 3$$

The stationary value is $\bar{Z} = \bar{z} = 9$, which needs to be tested against a second-order condition before we can tell whether it is a maximum or minimum (or neither). That will be taken up later.

Example 2 Find the extremum of

$$z = x_1^2 + x_2^2 \qquad \text{subject to} \qquad x_1 + 4x_2 = 2$$

The Lagrangian function is

$$Z = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$$

for which the necessary condition for a stationary value is

$$\left.\begin{aligned} Z_\lambda &= 2 - x_1 - 4x_2 = 0 \\ Z_1 &= 2x_1 - \lambda = 0 \\ Z_2 &= 2x_2 - 4\lambda = 0 \end{aligned}\right\} \qquad \text{or} \qquad \left\{\begin{aligned} x_1 + 4x_2 &= 2 \\ -\lambda + 2x_1 \qquad &= 0 \\ -4\lambda \qquad + 2x_2 &= 0 \end{aligned}\right.$$

The stationary value of Z, defined by the solution

$$\bar{\lambda} = \tfrac{4}{17} \qquad \bar{x}_1 = \tfrac{2}{17} \qquad \bar{x}_2 = \tfrac{8}{17}$$

is therefore $\bar{Z} = \bar{z} = \frac{4}{17}$. Again, a second-order condition should be consulted before we can tell whether $\bar{z}$ is a maximum or a minimum.

Total-Differential Approach

In the discussion of the free extremum of $z = f(x, y)$, it was learned that the first-order necessary condition may be stated in terms of the total differential dz as follows:

$$(12.9) \qquad dz = f_x\,dx + f_y\,dy = 0$$

This statement remains valid after a constraint $g(x, y) = c$ is added. However, with the constraint in the picture, we can no longer take dx and dy both as "arbitrary" changes as before. For if $g(x, y) = c$, then dg must be equal to dc, which is zero since c is a constant. Hence,

$$(12.10) \qquad (dg =)\, g_x\,dx + g_y\,dy = 0$$

and this relation makes dx and dy dependent on each other. The first-order necessary condition therefore becomes $dz = 0$ [(12.9)], subject to $g = c$, and hence also subject to $dg = 0$ [(12.10)]. By visual inspection of (12.9) and (12.10), it should be clear that, in order to satisfy this necessary condition, we must have

$$(12.11) \qquad \frac{f_x}{g_x} = \frac{f_y}{g_y}$$

This result can be verified by solving (12.10) for dy and substituting the result into (12.9). The condition (12.11), together with the constraint $g(x, y) = c$, will provide two equations from which to find the critical values of x and y.*

Does the total-differential approach yield the same first-order condition as the Lagrange-multiplier method? Let us compare (12.8) with the result just obtained. The first equation in (12.8) merely repeats the constraint; the new result requires its satisfaction also. The last two equations in (12.8) can be rewritten, respectively, as

$$(12.11') \qquad \frac{f_x}{g_x} = \lambda \qquad \text{and} \qquad \frac{f_y}{g_y} = \lambda$$

and these convey precisely the same information as (12.11). Note, however, that whereas the total-differential approach yields only the values of $\bar{x}$ and $\bar{y}$, the Lagrange-multiplier method also gives the value of $\bar{\lambda}$ as a direct by-product. As it turns out, $\bar{\lambda}$ provides a measure of the sensitivity of $\bar{Z}$ (and $\bar{z}$) to a shift of the constraint, as we shall presently demonstrate. Therefore, the Lagrange-multiplier

* Note that the constraint $g = c$ is still to be considered along with (12.11), even though we have utilized the equation $dg = 0$—that is, (12.10)—in deriving (12.11). While $g = c$ necessarily implies $dg = 0$, the converse is not true: $dg = 0$ merely implies $g =$ a constant (not necessarily c). Unless the constraint is explicitly considered, therefore, some information will be unwittingly left out of the problem.

method offers the advantage of containing certain built-in comparative-static information in the solution.

An Interpretation of the Lagrange Multiplier

To show that $\bar{\lambda}$ indeed measures the sensitivity of $\bar{Z}$ to changes in the constraint, let us perform a comparative-static analysis on the first-order condition (12.8). Since λ, x, and y are endogenous, the only available exogenous variable is the constraint parameter c. A change in c would cause a shift of the constraint curve in the xy plane and thereby alter the optimal solution. In particular, the effect of an *increase* in c (a larger budget, or a larger production quota) would indicate how the optimal solution is affected by a *relaxation* of the constraint.

To do the comparative-static analysis, we again resort to the implicit-function theorem. Taking the three equations in (12.8) to be in the form of $F^j(\lambda, x, y; c) = 0$ (with $j = 1, 2, 3$), and assuming them to have continuous partial derivatives, we must first check that the following endogenous-variable Jacobian (where $f_{xy} = f_{yx}$, and $g_{xy} = g_{yx}$)

$$(12.12) \qquad |J| = \begin{vmatrix} \frac{\partial F^1}{\partial \lambda} & \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial y} \\ \frac{\partial F^2}{\partial \lambda} & \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial y} \\ \frac{\partial F^3}{\partial \lambda} & \frac{\partial F^3}{\partial x} & \frac{\partial F^3}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{vmatrix}$$

does not vanish in the optimal state. At this moment, there is certainly no inkling that this would be the case. But our previous experience with the comparative statics of optimization problems [see the discussion of (11.42)] would suggest that this Jacobian is closely related to the second-order sufficient condition, and that if the sufficient condition is satisfied, then the Jacobian will be nonzero at the equilibrium (optimum). Leaving the full demonstration of this fact to the following section, let us proceed on the assumption that $|J| \neq 0$. If so, then we can express $\bar{\lambda}$, $\bar{x}$, and $\bar{y}$ all as implicit functions of the parameter c:

$$(12.13) \qquad \bar{\lambda} = \bar{\lambda}(c) \qquad \bar{x} = \bar{x}(c) \qquad \text{and} \qquad \bar{y} = \bar{y}(c)$$

all of which will have continuous derivatives. Also, we have the identities

$$(12.14) \qquad \begin{aligned} c - g(\bar{x}, \bar{y}) &\equiv 0 \\ f_x(\bar{x}, \bar{y}) - \bar{\lambda} g_x(\bar{x}, \bar{y}) &\equiv 0 \\ f_y(\bar{x}, \bar{y}) - \bar{\lambda} g_y(\bar{x}, \bar{y}) &\equiv 0 \end{aligned}$$

Now since the optimal value of Z depends on $\bar{\lambda}$, $\bar{x}$, and $\bar{y}$, that is,

$$(12.15) \qquad \bar{Z} = f(\bar{x}, \bar{y}) + \bar{\lambda}[c - g(\bar{x}, \bar{y})]$$

we may, in view of (12.13), consider $\bar{Z}$ to be a function of c alone. Differentiating

$\bar{Z}$ totally with respect to c, we find

$$\begin{aligned}\frac{d\bar{Z}}{dc} &= f_x \frac{d\bar{x}}{dc} + f_y \frac{d\bar{y}}{dc} + [c - g(\bar{x}, \bar{y})] \frac{d\bar{\lambda}}{dc} + \bar{\lambda}\left(1 - g_x \frac{d\bar{x}}{dc} - g_y \frac{d\bar{y}}{dc}\right) \\ &= (f_x - \bar{\lambda} g_x) \frac{d\bar{x}}{dc} + (f_y - \bar{\lambda} g_y) \frac{d\bar{y}}{dc} + [c - g(\bar{x}, \bar{y})] \frac{d\bar{\lambda}}{dc} + \bar{\lambda}\end{aligned}$$

where f_x, f_y, g_x, and g_y are all to be evaluated at the optimum. By (12.14), however, the first three terms on the right will all drop out. Thus we are left with the simple result

$$\frac{d\bar{Z}}{dc} = \bar{\lambda} \tag{12.16}$$

which validates our claim that the solution value of the Lagrange multiplier constitutes a measure of the effect of a change in the constraint via the parameter c on the optimal value of the objective function.

A word of caution, however, is perhaps in order here. For this interpretation of $\bar{\lambda}$, you must express Z specifically as in (12.7). In particular, write the last term as $\lambda[c - g(x, y)]$, *not* $\lambda[g(x, y) - c]$.

n-Variable and Multiconstraint Cases

The generalization of the Lagrange-multiplier method to n variables can be easily carried out if we write the choice variables in subscript notation. The objective function will then be in the form

$$z = f(x_1, x_2, \ldots, x_n)$$

subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = c$$

It follows that the Lagrangian function will be

$$Z = f(x_1, x_2, \ldots, x_n) + \lambda[c - g(x_1, x_2, \ldots, x_n)]$$

for which the first-order condition will consist of the following $(n + 1)$ simultaneous equations:

$$\begin{aligned}Z_\lambda &= c - g(x_1, x_2, \ldots, x_n) = 0 \\ Z_1 &= f_1 - \lambda g_1 = 0 \\ Z_2 &= f_2 - \lambda g_2 = 0 \\ &\cdots\cdots\cdots\cdots \\ Z_n &= f_n - \lambda g_n = 0\end{aligned}$$

Again, the first of these equations will assure us that the constraint is met, even though we are to focus our attention on the *free* Lagrangian function.

When there is more than one constraint, the Lagrange-multiplier method is equally applicable, provided we introduce as many such multipliers as there are constraints in the Lagrangian function. Let an n-variable function be subject simultaneously to the two constraints

$$g(x_1, x_2, \ldots, x_n) = c \qquad \text{and} \qquad h(x_1, x_2, \ldots, x_n) = d$$

Then, adopting λ and μ (the Greek letter mu) as the two undetermined multipliers, we may construct a Lagrangian function as follows:

$$Z = f(x_1, x_2, \ldots, x_n) + \lambda[c - g(x_1, x_2, \ldots, x_n)] + \mu[d - h(x_1, x_2, \ldots, x_n)]$$

This function will have the same value as the original objective function f if both constraints are satisfied, i.e., if the last two terms in the Lagrangian function both vanish. Considering λ and μ as variables, we now count $(n + 2)$ variables altogether; thus the first-order condition will in this case consist of the following $(n + 2)$ simultaneous equations:

$$Z_\lambda = c - g(x_1, x_2, \ldots, x_n) = 0$$

$$Z_\mu = d - h(x_1, x_2, \ldots, x_n) = 0$$

$$Z_i = f_i - \lambda g_i - \mu h_i = 0 \qquad (i = 1, 2, \ldots, n)$$

These should normally enable us to solve for all the x_i as well as λ and μ. As before, the first two equations of the necessary condition represent essentially a mere restatement of the two constraints.

EXERCISE 12.2

1 Use the Lagrange-multiplier method to find the stationary values of z:

(a) $z = xy$, subject to $x + 2y = 2$

(b) $z = x(y + 4)$, subject to $x + y = 8$

(c) $z = x - 3y - xy$, subject to $x + y = 6$

(d) $z = 7 - y + x^2$, subject to $x + y = 0$

2 In the above problem, find whether a slight relaxation of the constraint will increase or decrease the optimal value of z. At what rate?

3 Write the Lagrangian function and the first-order condition for stationary values (without solving the equations) for each of the following:

(a) $z = x + 2y + 3w + xy - yw$, subject to $x + y + 2w = 10$

(b) $z = x^2 + 2xy + yw^2$, subject to $2x + y + w^2 = 24$ and $x + w = 8$

4 If, instead of $g(x, y) = c$, the constraint is written in the form of $G(x, y) = 0$, how should the Lagrangian function and the first-order condition be modified as a consequence?

5 In discussing the total-differential approach, it was pointed out that, given the constraint $g(x, y) = c$, we may deduce that $dg = 0$. By the same token, we can further deduce that $d^2g = d(dg) = d(0) = 0$. Yet, in our earlier discussion of the unconstrained extremum of a function $z = f(x, y)$, we had a situation where $dz = 0$ is accompanied by either a positive definite or a negative definite d^2z, rather than $d^2z = 0$. How would you account for this disparity of treatment in the two cases?