CHAPTER

THIRTEEN

ECONOMIC DYNAMICS AND INTEGRAL CALCULUS

The term *dynamics*, as applied to economic analysis, has had different meanings at different times and for different economists.* In standard usage today, however, the term refers to the type of analysis in which the object is either to trace and study the specific time paths of the variables or to determine whether, given sufficient time, these variables will tend to converge to certain (equilibrium) values. This type of information is important because it fills a serious gap that marred our study of statics and comparative statics. In the latter, we always make the arbitrary assumption that the process of economic adjustment inevitably leads to an equilibrium. In a dynamic analysis, the question of "attainability" is to be squarely faced, rather than assumed away.

One salient feature of dynamic analysis is the *dating* of the variables, which introduces the explicit consideration of *time* into the picture. This can be done in two ways: time can be considered either as a *continuous* variable or as a *discrete* variable. In the former case, something is happening to the variable at each *point* of time (such as in continuous interest compounding); whereas in the latter, the variable undergoes a change only once within a *period* of time (e.g., interest is

* Fritz Machlup, "Statics and Dynamics: Kaleidoscopic Words," *Southern Economic Journal*, October, 1959, pp. 91–110; reprinted in Machlup, *Essays on Economic Semantics*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963, pp. 9–42.

added only at the end of every 6 months). One of these time concepts may be more appropriate than the other in certain contexts.

We shall discuss first the continuous-time case, to which the mathematical techniques of *integral calculus* and *differential equations* are pertinent. Later, in Chaps. 16 and 17, we shall turn to the discrete-time case, which utilizes the methods of *difference equations*.

13.1 DYNAMICS AND INTEGRATION

In a static model, generally speaking, the problem is to find the values of the endogenous variables that satisfy some specified equilibrium condition(s). Applied to the context of optimization models, the task becomes one of finding the values of the choice variables that maximize (or minimize) a specific objective function—with the first-order condition serving as the equilibrium condition. In a dynamic model, by contrast, the problem involves instead the delineation of the time path of some variable, on the basis of a known pattern of change (say, a given instantaneous rate of change).

An example should make this clear. Suppose that population size H is known to change over time at the rate

$$\frac{dH}{dt} = t^{-1/2} \tag{13.1}$$

We then try to find what time path(s) of population $H = H(t)$ can yield the rate of change in (13.1).

You will recognize that, if we know the function $H = H(t)$ to begin with, the derivative dH/dt can be found by differentiation. But in the problem now confronting us, the shoe is on the other foot: we are called upon to uncover the *primitive* function from a given *derived* function, rather than the reverse. Mathematically, we now need the exact opposite of the method of differentiation, or of differential calculus.

The relevant method, known as *integration*, or *integral calculus*, will be studied below. For the time being, let us be content with the observation that the function $H(t) = 2t^{1/2}$ does indeed have a derivative of the form in (13.1), thus apparently qualifying as a solution to our problem. The trouble is that there also exist similar functions, such as $H(t) = 2t^{1/2} + 15$ or $H(t) = 2t^{1/2} + 99$ or, more generally,

$$H(t) = 2t^{1/2} + c \qquad (c = \text{an arbitrary constant}) \tag{13.2}$$

which all possess exactly the same derivative (13.1). No unique time path can be determined, therefore, unless the value of the constant c can somehow be made definite. To accomplish this, additional information must be introduced into the model, usually in the form of what is known as an *initial condition* or *boundary condition*.

If we have knowledge of the initial population $H(0)$—that is, the value of H at $t = 0$, let us say, $H(0) = 100$—then the value of the constant c can be made determinate. Setting $t = 0$ in (13.2), we get

$$H(0) = 2(0)^{1/2} + c = c$$

But if $H(0) = 100$, then $c = 100$, and (13.2) becomes

$$H(t) = 2t^{1/2} + 100 \tag{13.2'}$$

where the constant is no longer arbitrary. More generally, for any given initial population $H(0)$, the time path will be

$$H(t) = 2t^{1/2} + H(0) \tag{13.2''}$$

Thus the population size H at any point of time will, in the present example, consist of the sum of the initial population $H(0)$ and another term involving the time variable t. Such a time path indeed charts the complete itinerary of the variable H over time, and thus it truly constitutes the solution to our dynamic model. [Equation (13.1) is also a function of t. Why can't *it* be considered a solution as well?]

Simple as it is, this population example illustrates the quintessence of the problems of economic dynamics. Given the pattern of behavior of a variable over time, we seek to find a function that describes the time path of the variable. In the process, we shall encounter one or more arbitrary constants, but if we possess sufficient additional information in the form of *initial conditions*, it will be possible to definitize these arbitrary constants.

In the simpler types of problem, such as the one cited above, the solution can be found by the method of integral calculus, which deals with the process of tracing a given derivative function back to its primitive function. In more complicated cases, we can also resort to the known techniques of the closely related branch of mathematics known as *differential equations*. Since a differential equation is defined as any equation containing differential or derivative expressions, (13.1) surely qualifies as one; consequently, by finding its solution, we have in fact already solved a differential equation, albeit an exceedingly simple one.

Let us now proceed to the study of the basic concepts of integral calculus. Since we discussed differential calculus with x (rather than t) as the independent variable, for the sake of symmetry we shall use x here, too. For convenience, however, we shall in the present discussion denote the primitive and derived functions by $F(x)$ and $f(x)$, respectively, rather than distinguish them by the use of a prime.

13.2 INDEFINITE INTEGRALS

The Nature of Integrals

It has been mentioned that integration is the reverse of differentiation. If differentiation of a given primitive function $F(x)$ yields the derivative $f(x)$, we can "integrate" $f(x)$ to find $F(x)$, provided appropriate information is available to definitize the arbitrary constant which will arise in the process of integration. The function $F(x)$ is referred to as an *integral* (or *antiderivative*) of the function $f(x)$. These two types of process may thus be likened to two ways of studying a family tree: *integration* involves the tracing of the parentage of the function $f(x)$, whereas *differentiation* seeks out the progeny of the function $F(x)$. But note this difference—while the (differentiable) primitive function $F(x)$ invariably produces a lone offspring, namely, a unique derivative $f(x)$, the derived function $f(x)$ is traceable to an infinite number of possible parents through integration, because if $F(x)$ is an integral of $f(x)$, then so also must be $F(x)$ plus any constant, as we saw in (13.2).

We need a special notation to denote the required integration of $f(x)$ with respect to x. The standard one is

$$\int f(x)\,dx$$

The symbol on the left—an elongated S (with the connotation of sum, to be explained later)—is called the *integral sign*, whereas the $f(x)$ part is known as the *integrand* (the function to be integrated), and the dx part—similar to the dx in the differentiation operator d/dx—reminds us that the operation is to be performed with respect to the variable x. However, you may also take $f(x)\,dx$ as a single entity and interpret it as the differential of the primitive function $F(x)$ [that is, $dF(x) = f(x)\,dx$]. Then, the integral sign in front can be viewed as an instruction to reverse the differentiation process that gave rise to the differential. With this new notation, we can write that

$$(13.3) \qquad \frac{d}{dx}F(x) = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + c$$

where the presence of c, an arbitrary *constant of integration*, serves to indicate the multiple parentage of the integrand.

The integral $\int f(x)\,dx$ is, more specifically, known as the *indefinite integral* of $f(x)$ (as against the *definite integral* to be discussed in the next section), because it has no definite numerical value. Because it is equal to $F(x) + c$, its value will in general vary with the value of x (even if c is definitized). Thus, like a derivative, an indefinite integral is itself a function of the variable x.

Basic Rules of Integration

Just as there are rules of derivation, we can also develop certain rules of integration. As may be expected, the latter are heavily dependent on the rules of derivation with which we are already familiar. From the following derivative formula for a power function,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad (n \neq -1)$$

for instance, we see that the expression $x^{n+1}/(n+1)$ is the primitive function for the derivative function x^n; thus, by substituting these for $F(x)$ and $f(x)$ in (13.3), we may state the result as a rule of integration.

Rule I (the power rule)

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

Example 1 Find $\int x^3 \, dx$. Here, we have $n = 3$, and therefore

$$\int x^3 \, dx = \frac{1}{4}x^4 + c$$

Example 2 Find $\int x \, dx$. Since $n = 1$, we have

$$\int x \, dx = \frac{1}{2}x^2 + c$$

Example 3 What is $\int 1 \, dx$? To find this integral, we recall that $x^0 = 1$; so we can let $n = 0$ in the power rule and get

$$\int 1 \, dx = x + c$$

[$\int 1 \, dx$ is sometimes written simply as $\int dx$, since $1 \, dx = dx$.]

Example 4 Find $\int \sqrt{x^3} \, dx$. Since $\sqrt{x^3} = x^{3/2}$, we have $n = \frac{3}{2}$; therefore,

$$\int \sqrt{x^3} \, dx = \frac{x^{5/2}}{\frac{5}{2}} + c = \frac{2}{5}\sqrt{x^5} + c$$

Example 5 Find $\int \frac{1}{x^4} dx$, $(x \neq 0)$. Since $1/x^4 = x^{-4}$, we have $n = -4$. Thus the integral is

$$\int \frac{1}{x^4} dx = \frac{x^{-4+1}}{-4+1} + c = -\frac{1}{3x^3} + c$$

Note that the correctness of the results of integration can always be checked by differentiation; if the integration is correct, the derivative of the integral must be equal to the integrand.

The derivative formulas for simple exponential and logarithmic functions have been shown to be

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\ln x = \frac{1}{x} \qquad (x > 0)$$

From these, two other basic rules of integration emerge.

Rule II (the exponential rule)

$$\int e^x \, dx = e^x + c$$

Rule III (the logarithmic rule)

$$\int \frac{1}{x}\,dx = \ln x + c \qquad (x > 0)$$

It is of interest that the integrand involved in Rule III is $1/x = x^{-1}$, which is a special form of the power function x^n with $n = -1$. This particular integrand is inadmissible under the power rule, but now is duly taken care of by the logarithmic rule.

As stated, the logarithmic rule is placed under the restriction $x > 0$, because logarithms do not exist for nonpositive values of x. A more general formulation of the rule, which can take care of negative values of x, is

$$\int \frac{1}{x}\,dx = \ln |x| + c \qquad (x \neq 0)$$

which also implies that $(d/dx)\ln |x| = 1/x$, just as $(d/dx)\ln x = 1/x$. You should convince yourself that the replacement of x (with the restriction $x > 0$) by $|x|$ (with the restriction $x \neq 0$) does not vitiate the formula in any way.

Also, as a matter of notation, it should be pointed out that the integral $\int \frac{1}{x}\,dx$ is sometimes also written as $\int \frac{dx}{x}$.

As variants of Rules II and III, we also have the following two rules.

Rule IIa

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

Rule IIIa

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c \qquad [f(x) > 0]$$

$$\text{or} \qquad \ln |f(x)| + c \qquad [f(x) \neq 0]$$

The bases for these two rules can be found in the derivative rules in (10.20).

Rules of Operation

The three rules given above amply illustrate the spirit underlying all rules of integration. Each rule always corresponds to a certain derivative formula. Also, an arbitrary constant is always appended at the end (even though it is to be definitized later by using a given boundary condition) to indicate that a whole family of primitive functions can give rise to the given integrand.

To be able to deal with more complicated integrands, however, we shall also find the following two rules of operation with regard to integrals helpful.

Rule IV (the integral of a sum) The integral of the sum of a finite number of functions is the sum of the integrals of those functions. For the two-function case, this means that

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

This rule is a natural consequence of the fact that

$$\underbrace{\frac{d}{dx}[F(x) + G(x)]}_{A} = \underbrace{\frac{d}{dx}F(x) + \frac{d}{dx}G(x)}_{B} = \underbrace{f(x) + g(x)}_{C}$$

Inasmuch as $A = C$, on the basis of (13.3) we can write

$$\int [f(x) + g(x)]\, dx = F(x) + G(x) + c \tag{13.4}$$

But, from the fact that $B = C$, it follows that

$$\int f(x)\, dx = F(x) + c_1 \qquad \text{and} \qquad \int g(x)\, dx = G(x) + c_2$$

Thus we can obtain (by addition)

$$\int f(x)\, dx + \int g(x)\, dx = F(x) + G(x) + c_1 + c_2 \tag{13.5}$$

Since the constants c, c_1, and c_2 are arbitrary in value, we can let $c = c_1 + c_2$. Then the right sides of (13.4) and (13.5) become equal, and as a consequence, their left sides must be equal also. This proves Rule IV.

Example 6 Find $\int (x^3 + x + 1)\, dx$. By Rule IV, this integral can be expressed as a sum of three integrals: $\int x^3\, dx + \int x\, dx + \int 1\, dx$. Since the values of these three integrals have previously been found in Examples 1, 2, and 3, we can simply combine those results to get

$$\int (x^3 + x + 1)\, dx = \left(\frac{x^4}{4} + c_1\right) + \left(\frac{x^2}{2} + c_2\right) + (x + c_3)$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + x + c$$

In the final answer, we have lumped together the three subscripted constants into a single constant c.

As a general practice, all the additive arbitrary constants of integration that emerge during the process can always be combined into a single arbitrary constant in the final answer.

Example 7 Find $\int \left(2e^{2x} + \frac{14x}{7x^2 + 5}\right) dx$. By Rule IV, we can integrate the two additive terms in the integrand separately, and then sum the results. Since the

$2e^{2x}$ term is in the format of $f'(x)e^{f(x)}$ in Rule IIa, with $f(x) = 2x$, the integral is $e^{2x} + c_1$. Similarly, the other term, $14x/(7x^2 + 5)$, takes the form of $f'(x)/f(x)$, with $f(x) = 7x^2 + 5 > 0$. Thus, by Rule IIIa, the integral is $\ln(7x^2 + 5) + c_2$. Hence we can write

$$\int \left(2e^{2x} + \frac{14x}{7x^2 + 5}\right) dx = e^{2x} + \ln(7x^2 + 5) + c$$

where we have combined c_1 and c_2 into one arbitrary constant c.

Rule V (the integral of a multiple) The integral of k times an integrand (k being a constant) is k times the integral of that integrand. In symbols,

$$\int kf(x)\, dx = k\int f(x)\, dx$$

What this rule amounts to, operationally, is that a multiplicative constant can be "factored out" of the integral sign. (*Warning:* A *variable* term *cannot* be factored out in this fashion!) To prove this rule, we recall that k times $f(x)$ merely means adding $f(x)$ k times; therefore, by Rule IV,

$$\int kf(x)\, dx = \int \underbrace{[f(x) + f(x) + \cdots + f(x)]}_{k \text{ terms}}\, dx$$

$$= \underbrace{\int f(x)\, dx + \int f(x)\, dx + \cdots + \int f(x)\, dx}_{k \text{ terms}} = k\int f(x)\, dx$$

Example 8 Find $\int -f(x)\, dx$. Here $k = -1$, and thus

$$\int -f(x)\, dx = -\int f(x)\, dx$$

That is, the integral of the negative of a function is the negative of the integral of that function.

Example 9 Find $\int 2x^2\, dx$. Factoring out the 2 and applying Rule I, we have

$$\int 2x^2\, dx = 2\int x^2\, dx = 2\left(\frac{x^3}{3} + c_1\right) = \frac{2}{3}x^3 + c$$

Example 10 Find $\int 3x^2\, dx$. In this case, factoring out the multiplicative constant yields

$$\int 3x^2\, dx = 3\int x^2\, dx = 3\left(\frac{x^3}{3} + c_1\right) = x^3 + c$$

Note that, in contrast to the preceding example, the term x^3 in the final answer does not have any fractional expression attached to it. This neat result is due to the fact that 3 (the multiplicative constant of the integrand) happens to be precisely equal to 2 (the power of the function) plus 1. Referring to the power rule (Rule I), we see that the multiplicative constant $(n + 1)$ will in such a case cancel out the fraction $1/(n + 1)$, thereby yielding $(x^{n+1} + c)$ as the answer.

In general, whenever we have an expression $(n + 1)x^n$ as the integrand, there is really no need to factor out the constant $(n + 1)$ and then integrate x^n; instead, we may write $x^{n+1} + c$ as the answer right away.

Example 11 Find $\int \left(5e^x - x^{-2} + \frac{3}{x}\right)dx$, $(x \neq 0)$. This example illustrates both Rules IV and V; actually, it illustrates the first three rules as well:

$$\int \left(5e^x - \frac{1}{x^2} + \frac{3}{x}\right) dx = 5\int e^x\,dx - \int x^{-2}\,dx + 3\int \frac{1}{x}\,dx \qquad \text{[by Rules IV and V]}$$

$$= (5e^x + c_1) - \left(\frac{x^{-1}}{-1} + c_2\right) + (3\ln|x| + c_3)$$

$$= 5e^x + \frac{1}{x} + 3\ln|x| + c$$

The correctness of the result can again be verified by differentiation.

Rules Involving Substitution

Now we shall introduce two more rules of integration which seek to simplify the process of integration, when the circumstances are appropriate, by a substitution of the original variable of integration. Whenever the newly introduced variable of integration makes the integration process easier than under the old, these rules will become of service.

Rule VI (the substitution rule) The integral of $f(u)(du/dx)$ with respect to the variable x is the integral of $f(u)$ with respect to the variable u:

$$\int f(u)\frac{du}{dx}dx = \int f(u)\,du = F(u) + c$$

where the operation $\int du$ has been substituted for the operation $\int dx$.

This rule, the integral-calculus counterpart of the chain rule, may be proved by means of the chain rule itself. Given a function $F(u)$, where $u = u(x)$, the chain rule states that

$$\frac{d}{dx}F(u) = \frac{d}{du}F(u)\frac{du}{dx} = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}$$

Since $f(u)(du/dx)$ is the derivative of $F(u)$, it follows from (13.3) that the integral (antiderivative) of the former must be

$$\int f(u)\frac{du}{dx}dx = F(u) + c$$

You may note that this result, in fact, follows also from the *canceling* of the two dx expressions on the left.

Example 12 Find $\int 2x(x^2 + 1)\,dx$. The answer to this can be obtained by first multiplying out the integrand:

$$\int 2x(x^2 + 1)\,dx = \int (2x^3 + 2x)\,dx = \frac{x^4}{2} + x^2 + c$$

but let us now do it by the substitution rule. Let $u = x^2 + 1$; then $du/dx = 2x$, or $dx = du/2x$. Substitution of $du/2x$ for dx will yield

$$\int 2x(x^2 + 1)\,dx = \int 2xu\frac{du}{2x} = \int u\,du = \frac{u^2}{2} + c_1$$

$$= \frac{1}{2}(x^4 + 2x^2 + 1) + c_1 = \frac{1}{2}x^4 + x^2 + c$$

where $c = \frac{1}{2} + c_1$. The same answer can also be obtained by substituting du/dx for $2x$ (instead of $du/2x$ for dx).

Example 13 Find $\int 6x^2(x^3 + 2)^{99}\,dx$. The integrand of this example is not easily multiplied out, and thus the substitution rule now has a better opportunity to display its effectiveness. Let $u = x^3 + 2$; then $du/dx = 3x^2$, so that

$$\int 6x^2(x^3 + 2)^{99}\,dx = \int\left(2\frac{du}{dx}\right)u^{99}\,dx = \int 2u^{99}\,du$$

$$= \frac{2}{100}u^{100} + c = \frac{1}{50}(x^3 + 2)^{100} + c$$

Example 14 Find $\int 8e^{2x+3}\,dx$. Let $u = 2x + 3$; then $du/dx = 2$, or $dx = du/2$. Hence,

$$\int 8e^{2x+3}\,dx = \int 8e^u\frac{du}{2} = 4\int e^u\,du = 4e^u + c = 4e^{2x+3} + c$$

As these examples show, this rule is of help whenever we can—by the judicious choice of a function $u = u(x)$—express the integrand (a function of x) as the product of $f(u)$ (a function of u) and du/dx (the derivative of the u function which we have chosen). However, as illustrated by the last two examples, this rule can be used also when the original integrand is transformable into a constant multiple of $f(u)(du/dx)$. This would not affect the applicability because the constant multiplier can be factored out of the integral sign, which would then leave an integrand of the form $f(u)(du/dx)$, as required in the substitution rule. When the substitution of variables results in a *variable* multiple of $f(u)(du/dx)$, say, x times the latter, however, factoring is not permissible, and this rule will be of no help. In fact, there exists no general formula giving the integral of a product of two functions in terms of the separate integrals of those functions; nor do we have a general formula giving the integral of a quotient of two functions in terms of their separate integrals. Herein lies the reason why integration, on the whole, is more difficult than differentiation and why, with complicated integrands, it is more convenient to look up the answer in prepared tables of integration formulas rather than to undertake the integration by oneself.

Rule VII (integration by parts) The integral of v with respect to u is equal to uv less the integral of u with respect to v:

$$\int v\,du = uv - \int u\,dv$$

The essence of this rule is to replace the operation $\int du$ by the operation $\int dv$.

The rationale behind this result is relatively simple. First, the product rule of differentials gives us

$$d(uv) = v\,du + u\,dv$$

If we integrate both sides of the equation (i.e., integrate each differential), we get a new equation

$$\int d(uv) = \int v\,du + \int u\,dv$$

$$\text{or} \qquad uv = \int v\,du + \int u\,dv \qquad \text{[no constant is needed on the left (why?)]}$$

Then, by subtracting $\int u\,dv$ from both sides, the result stated above emerges.

Example 15 Find $\int x(x+1)^{1/2}\,dx$. Unlike Examples 12 and 13, the present example is not amenable to the type of substitution used in Rule VI. (Why?) However, we may consider the given integral to be in the form of $\int v\,du$, and apply Rule VII. To this end, we shall let $v = x$, implying $dv = dx$, and also let $u = \frac{2}{3}(x+1)^{3/2}$, so that $du = (x+1)^{1/2}\,dx$. Then we can find the integral to be

$$\begin{aligned}\int x(x+1)^{1/2}\,dx &= \int v\,du = uv - \int u\,dv \\ &= \frac{2}{3}(x+1)^{3/2}x - \int \frac{2}{3}(x+1)^{3/2}\,dx \\ &= \frac{2}{3}(x+1)^{3/2}x - \frac{4}{15}(x+1)^{5/2} + c\end{aligned}$$

Example 16 Find $\int \ln x\,dx$, $(x > 0)$. We cannot apply the logarithmic rule here, because that rule deals with the integrand $1/x$, not $\ln x$. Nor can we use Rule VI. But if we let $v = \ln x$, implying $dv = (1/x)\,dx$, and also let $u = x$, so that

$du = dx$, then the integration can be performed as follows:

$$\int \ln x \, dx = \int v \, du = uv - \int u \, dv$$
$$= x \ln x - \int dx = x \ln x - x + c = x(\ln x - 1) + c$$

Example 17 Find $\int xe^x \, dx$. In this case, we shall simply let $v = x$, and $u = e^x$, so that $dv = dx$ and $du = e^x \, dx$. Applying Rule VII, we then have

$$\int xe^x \, dx = \int v \, du = uv - \int u \, dv$$
$$= e^x x - \int e^x \, dx = e^x x - e^x + c = e^x(x - 1) + c$$

The validity of this result, like those of the preceding examples, can of course be readily checked by differentiation.

EXERCISE 13.2

1 Find the following:

(*a*) $\int 16x^{-3} \, dx \quad (x \neq 0)$

(*b*) $\int 9x^8 \, dx$

(*c*) $\int (x^5 - 3x) \, dx$

(*d*) $\int 2e^{-2x} \, dx$

(*e*) $\int \frac{4x}{x^2 + 1} \, dx$

(*f*) $\int (2ax + b)(ax^2 + bx)^7 \, dx$

2 Find:

(*a*) $\int 13e^x \, dx$

(*b*) $\int \left(3e^x + \frac{4}{x}\right) dx \quad (x > 0)$

(*c*) $\int \left(5e^x + \frac{3}{x^2}\right) dx \quad (x \neq 0)$

(*d*) $\int 3e^{-(2x+7)} \, dx$

(*e*) $\int 4xe^{x^2+3} \, dx$

(*f*) $\int xe^{x^2+9} \, dx$

3 Find:

(*a*) $\int \frac{3dx}{x} \quad (x \neq 0)$

(*b*) $\int \frac{dx}{x - 2} \quad (x \neq 2)$

(*c*) $\int \frac{2x}{x^2 + 3} \, dx$

(*d*) $\int \frac{x}{3x^2 + 5} \, dx$

4 Find:

(*a*) $\int (x + 3)(x + 1)^{1/2} \, dx$

(*b*) $\int x \ln x \, dx \quad (x > 0)$

5 Given n constants k_i (with $i = 1, 2, \ldots, n$) and n functions $f_i(x)$, deduce from Rules IV and V that

$$\int \sum_{i=1}^{n} k_i f_i(x) \, dx = \sum_{i=1}^{n} k_i \int f_i(x) \, dx$$

13.3 DEFINITE INTEGRALS

Meaning of Definite Integrals

All the integrals cited in the preceding section are of the *indefinite* variety: each is a function of a variable and, hence, possesses no definite numerical value. Now, for a given indefinite integral of a continuous function $f(x)$,

$$\int f(x)\,dx = F(x) + c$$

if we choose two values of x in the domain, say, a and b $(a < b)$, substitute them successively into the right side of the equation, and form the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

we get a specific numerical value, free of the variable x as well as the arbitrary constant c. This value is called the *definite integral* of $f(x)$ from a to b. We refer to a as the *lower limit of integration* and to b as the *upper limit of integration*.

In order to indicate the limits of integration, we now modify the integral sign to the form $\int_a^b$. The evaluation of the definite integral is then symbolized in the following steps:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a) \tag{13.6}$$

where the symbol $]_a^b$ (also written $|_a^b$ or $[\cdots]_a^b$) is an instruction to substitute b and a, successively, for x in the result of integration to get $F(b)$ and $F(a)$, and then take their difference, as indicated on the right of (13.6). As the first step, however, we must find the indefinite integral, although we may omit the constant c, since the latter will drop out in the process of difference-taking anyway.

Example 1 Evaluate $\int_1^5 3x^2\,dx$. Since the indefinite integral is $x^3 + c$, this definite integral has the value

$$\int_1^5 3x^2\,dx = x^3\Big]_1^5 = (5)^3 - (1)^3 = 125 - 1 = 124$$

Example 2 Evaluate $\int_a^b ke^x\,dx$. Here, the limits of integration are given in symbols; consequently, the result of integration is also in terms of those symbols:

$$\int_a^b ke^x\,dx = ke^x\Big]_a^b = k(e^b - e^a)$$

Example 3 Evaluate $\int_0^4 \left(\frac{1}{1+x} + 2x \right) dx$, $(x \neq -1)$. The indefinite integral is $\ln|1+x| + x^2 + c$; thus the answer is

$$\int_0^4 \left(\frac{1}{1+x} + 2x \right) dx = \left[\ln|1+x| + x^2 \right]_0^4$$
$$= (\ln 5 + 16) - (\ln 1 + 0)$$
$$= \ln 5 + 16 \qquad [\text{since } \ln 1 = 0]$$

It is important to realize that the limits of integration a and b both refer to values of the variable x. Were we to use the substitution-of-variables technique (Rules VI and VII) during integration and introduce a variable, u, care should be taken *not* to consider a and b as the limits of u. The next example will illustrate this point.

Example 4 Evaluate $\int_1^2 (2x^3 - 1)^2 (6x^2)\, dx$. Let $u = 2x^3 - 1$; then $du/dx = 6x^2$, or $du = 6x^2\, dx$. Now notice that, when $x = 1$, u will be 1 but that, when $x = 2$, u will be 15; in other words, the limits of integration in terms of the variable u should be 1 (lower) and 15 (upper). Rewriting the given integral in u will therefore give us not $\int_1^2 u^2\, du$ but

$$\int_1^{15} u^2\, du = \frac{1}{3}u^3 \Big]_1^{15} = \frac{1}{3}(15^3 - 1^3) = 1124\tfrac{2}{3}$$

Alternatively, we may first convert u back to x and then use the original limits of 1 and 2 to get the identical answer:

$$\left[\frac{1}{3}u^3 \right]_{u=1}^{u=15} = \left[\frac{1}{3}(2x^3 - 1)^3 \right]_{x=1}^{x=2} = \frac{1}{3}(15^3 - 1^3) = 1124\tfrac{2}{3}$$

A Definite Integral as an Area Under a Curve

Every definite integral has a definite value. That value may be interpreted geometrically to be a particular area under a given curve.

The graph of a continuous function $y = f(x)$ is drawn in Fig. 13.1. If we seek to measure the (shaded) area A enclosed by the curve and the x axis between the two points a and b in the domain, we may proceed in the following manner. First, we divide the interval $[a, b]$ into n subintervals (not necessarily equal in length). Four of these are drawn in diagram a—that is, $n = 4$—the first being $[x_1, x_2]$ and the last, $[x_4, x_5]$. Since each of these represents a change in x, we may refer to them as $\Delta x_1, \ldots, \Delta x_4$, respectively. Now, on the subintervals let us construct four rectangular blocks such that the height of each block is equal to the highest value of the function attained in that block (which happens to occur at the left-side boundary of each rectangle here). The first block thus has the height $f(x_1)$ and the width Δx_1, and, in general, the ith block has the height $f(x_i)$ and the width Δx_i.

The total area A^* of this set of blocks is the sum

$$A^* = \sum_{i=1}^{n} f(x_i)\,\Delta x_i \qquad (n = 4 \text{ in Fig. 13.1}a)$$

This, though, is obviously *not* the area under the curve we seek, but only a very rough approximation thereof.

What makes A^* deviate from the true value of A is the unshaded portion of the rectangular blocks; these make A^* an *over*estimate of A. If the unshaded portion can be shrunk in size and be made to approach zero, however, the approximation value A^* will correspondingly approach the true value A. This result will materialize when we try a finer and finer segmentation of the interval $[a, b]$, so that n is increased and Δx_i is shortened indefinitely. Then the blocks will become more slender (if more numerous), and the protrusion beyond the curve will diminish, as can be seen in diagram b. Carried to the limit, this "slenderizing"

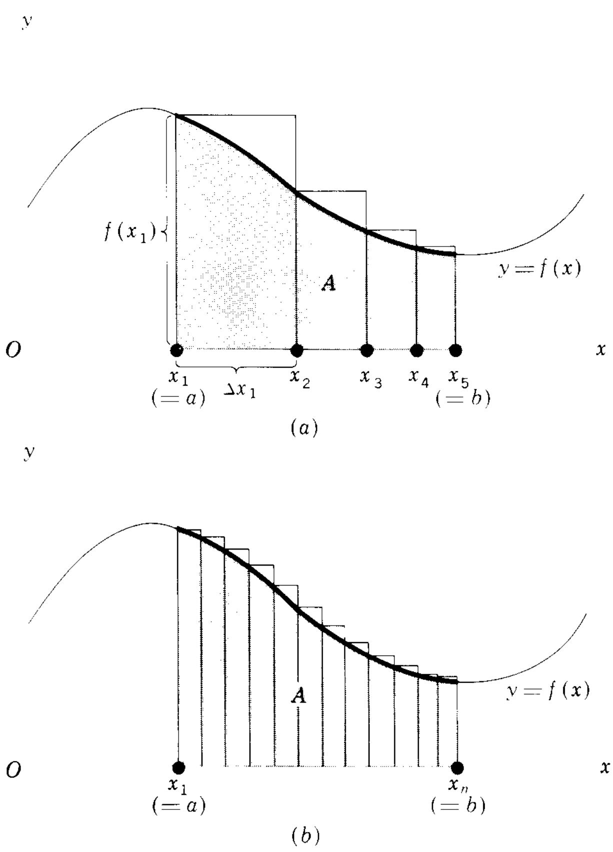

Figure 13.1

operation yields

$$(13.7)\qquad \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x_i = \lim_{n\to\infty} A^* = \text{area } A$$

provided this limit exists. (It does in the present case.) This equation, indeed, constitutes the formal definition of an area under a curve.

The summation expression in (13.7), $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$, bears a certain resemblance to the definite integral expression $\int_a^b f(x)\,dx$. Indeed, the latter is based on the former. When the change Δx_i is infinitesimal, we may replace it with the symbol dx_i. Moreover, the subscript i may be dropped because each of these infinitesimal changes can be equally well represented by the symbol dx. Thus we may rewrite $f(x_i)\,\Delta x_i$ into $f(x)\,dx$. What about the summation sign? The $\sum_{i=1}^{n}$ notation represents the sum of a *finite* number of terms. When we let $n \to \infty$, and take the limit of that sum, the regular notation for such an operation is rather cumbersome. Thus a simpler substitute is needed. That substitute is $\int_a^b$, where the elongated S symbol also indicates a sum, and where a and b (just as $i = 1$ and n) serve to specify the lower and upper limits of this sum. In short the definite integral is a shorthand for the limit-of-a-sum expression in (13.7). That is,

$$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x_i = \text{area } A$$

Thus the said definite integral (referred to as a *Riemann integral*) now has an *area* connotation as well as a *sum* connotation, because $\int_a^b$ is the continuous counterpart of the discrete concept of $\sum_{i=1}^{n}$.

In Fig. 13.1, we attempted to approximate area A by systematically reducing an *over*estimate A^* by finer segmentation of the interval $[a, b]$. The resulting limit of the sum of block areas is called the *upper integral*—an approximation from above. We could also have approximated area A from below by forming rectangular blocks inscribed by the curve rather than protruding beyond it (see Exercise 13.3-3). The total area A^{**} of this new set of blocks will *under*estimate A, but as the segmentation of $[a, b]$ becomes finer and finer, we shall again find $\lim_{n\to\infty} A^{**} = A$. The last-cited limit of the sum of block areas is called the *lower integral*. If, and only if, the upper integral and lower integral are equal in value, then the Riemann integral $\int_a^b f(x)\,dx$ is defined, and the function $f(x)$ is said to be *Riemann integrable*. There exist theorems specifying the conditions under which a function $f(x)$ is integrable. According to the fundamental theorem of calculus, a function is integrable in $[a, b]$ if it is continuous in that interval. As long as we

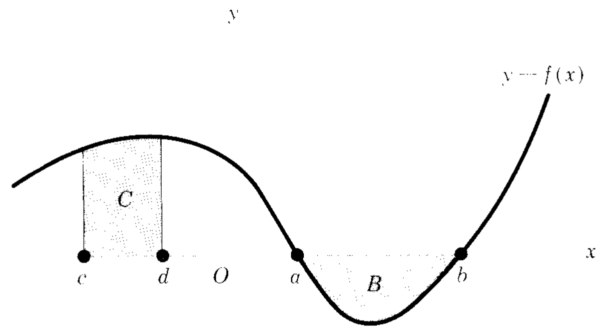

Figure 13.2

are working with continuous functions, therefore, we should have no worries in this regard.

Another point may be noted. Although the area A in Fig 13.1 happens to lie entirely under a decreasing portion of the curve $y = f(x)$, the conceptual equating of a definite integral with an area is valid also for upward-sloping portions of the curve. In fact, both types of slope may be present simultaneously; e.g., we can calculate $\int_0^b f(x)\,dx$ as the area under the curve in Fig. 13.1 above the line Ob.

Note that, if we calculate the area B in Fig. 13.2 by the definite integral $\int_a^b f(x)\,dx$, the answer will come out negative, because the height of each rectangular block involved in this area is negative. This gives rise to the notion of a *negative area*, an area that lies *below* the x axis and *above* a given curve. In case we are interested in the numerical rather than the algebraic value of such an area, therefore, we should take the absolute value of the relevant definite integral. The area $C = \int_c^d f(x)\,dx$, on the other hand, has a positive sign even though it lies in the negative region of the x axis; this is because each rectangular block has a positive height as well as a positive width when we are moving from c to d. From this, the implication is clear that interchange of the two limits of integration would, by reversing the direction of movement, alter the sign of Δx_i and of the definite integral. Applied to area B, we see that the definite integral $\int_b^a f(x)\,dx$ (from b to a) will give the negative of the area B; this will measure the numerical value of this area.

Some Properties of Definite Integrals

The discussion in the preceding paragraph leads us to the following property of definite integrals.

Property I The interchange of the limits of integration changes the sign of the definite integral:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

This can be proved as follows:

$$\int_b^a f(x)\,dx = F(a) - F(b) = -\,[F(b) - F(a)] = -\int_a^b f(x)\,dx$$

Definite integrals also possess some other interesting properties.

Property II A definite integral has a value of zero when the two limits of integration are identical:

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0$$

Under the "area" interpretation, this means that the area (under a curve) above any single *point* in the domain is nil. This is as it should be, because on top of a point on the x axis, we can draw only a (one-dimensional) *line*, never a (two-dimensional) *area*.

Property III A definite integral can be expressed as a sum of a finite number of definite subintegrals as follows:

$$\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx \qquad (a < b < c < d)$$

Only three subintegrals are shown in this equation, but the extension to the case of n subintegrals is also valid. This property is sometimes described as the *additivity property*.

In terms of area, this means that the area (under the curve) lying above the interval $[a, d]$ on the x axis can be obtained by summing the areas lying above the subintervals in the set $\{[a, b], [b, c], [c, d]\}$. Note that, since we are dealing with closed intervals, the border points b and c have each been included in *two* areas. Is this not double counting? It indeed is. But fortunately no damage is done, because by Property II the area above a single point is zero, so that the double counting produces no effect on the calculation. But, needless to say, the double counting of any *interval* is never permitted.

Earlier, it was mentioned that all continuous functions are Riemann integrable. Now, by Property III, we can also find the definite integrals (areas) of certain discontinuous functions. Consider the step function in Fig. 13.3a. In spite of the discontinuity at point b in the interval $[a, c]$, we can find the shaded area from the sum

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

The same also applies to the curve in diagram b.

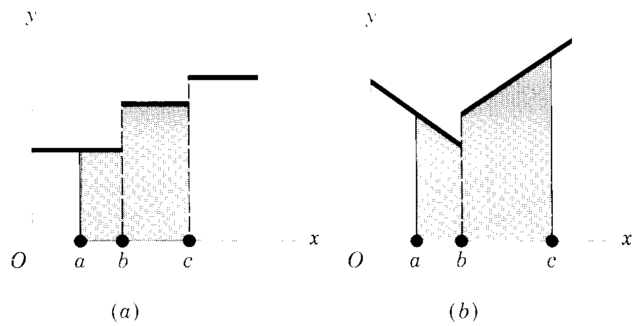

Figure 13.3

Property IV

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

Property V

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

Property VI

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Property VII (integration by parts) Given $u(x)$ and $v(x)$,

$$\int_{x=a}^{x=b} v\,du = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} u\,dv$$

These last four properties, all borrowed from the rules of indefinite integration, should require no further explanation.

Another Look at the Indefinite Integral

We introduced the definite integral by way of attaching two limits of integration to an indefinite integral. Now that we know the meaning of the definite integral, let us see how we can revert from the latter to the indefinite integral.

Suppose that, instead of fixing the upper limit of integration at b, we allow it to be a variable, designated simply as x. Then the integral will take the form

$$\int_a^x f(x)\,dx = F(x) - F(a)$$

which, now being a function of x, denotes a *variable* area under the curve of $f(x)$. But since the last term on the right is a constant, this integral must be a member of the family of primitive functions of $f(x)$, which we denoted earlier as $F(x) + c$. If we set $c = -F(a)$, then the above integral becomes exactly the indefinite integral $\int f(x)\,dx$.

From this point of view, therefore, we may consider the $\int$ symbol to mean the same as $\int_a^x$, provided it is understood that in the latter version of the symbol the lower limit of integration is related to the constant of integration by the equation $c = -F(a)$.

EXERCISE 13.3

1 Evaluate the following:

(a) $\int_1^3 \frac{1}{2}x^2\,dx$ (d) $\int_2^4 (x^3 - 6x^2)\,dx$

(b) $\int_0^1 x(x^2 + 6)\,dx$ (e) $\int_{-1}^1 (ax^2 + bx + c)\,dx$

(c) $\int_1^3 3\sqrt{x}\,dx$ (f) $\int_4^2 x^2\left(\frac{1}{3}x^3 + 1\right)dx$

2 Evaluate the following:

(a) $\int_1^2 e^{-2x}\,dx$ (c) $\int_2^3 (e^{2x} + e^x)\,dx$

(b) $\int_{-1}^{e-2} \frac{dx}{x+2}$ (d) $\int_e^6 \left(\frac{1}{x} + \frac{1}{1+x}\right)dx$

3 In Fig. 13.1*a*, take the lowest value of the function attained in each subinterval as the height of the rectangular block, i.e., take $f(x_2)$ instead of $f(x_1)$ as the height of the first block, though still retaining Δx_1 as its width, and do likewise for the other blocks.

(a) Write a summation expression for the total area A^{**} of the new rectangles.

(b) Does A^{**} overestimate or underestimate the desired area A?

(c) Would A^{**} tend to approach or to deviate further from A if a finer segmentation of $[a, b]$ were introduced? (*Hint:* Try a diagram.)

(d) In the limit, when the number n of subintervals approaches ∞, would the approximation value A^{**} approach the true value A, just as the approximation value A^* did?

(e) What can you conclude from the above about the Riemann integrability of the function $f(x)$ in the figure?

4 The definite integral $\int_a^b f(x)\,dx$ is said to represent an area under a curve. Does this curve refer to the graph of the integrand $f(x)$, or of the primitive function $F(x)$? If we plot the graph of the $F(x)$ function, how can we show the above definite integral on it—by an area, a line segment, or a point?

5 Verify that a constant c can be equivalently expressed as a definite integral:

(a) $c \equiv \int_0^b \frac{c}{b}\,dx$ (b) $c \equiv \int_0^c 1\,dt$

13.4 IMPROPER INTEGRALS

Certain integrals are said to be "improper." We shall briefly discuss two varieties thereof.

Infinite Limits of Integration

When we have definite integrals of the form

$$\int_a^{\infty} f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^{b} f(x)\,dx$$

with one limit of integration being infinite, we refer to them as *improper integrals*. In these cases, it is not possible to evaluate the integrals as, respectively,

$$F(\infty) - F(a) \qquad \text{and} \qquad F(b) - F(-\infty)$$

because ∞ is not a number, and therefore it cannot be substituted for x in the function $F(x)$. Instead, we must resort once more to the concept of limits.

The first improper integral cited above can be defined to be the limit of another (proper) integral as the latter's upper limit of integration tends to ∞; that is,

$$(13.8) \qquad \int_a^{\infty} f(x)\,dx \equiv \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If this limit exists, the improper integral is said to be convergent (or to converge), and the limiting process will yield the value of the integral. If the limit does not exist, the improper integral is said to be divergent and is in fact meaningless. By the same token, we can define

$$(13.8') \qquad \int_{-\infty}^{b} f(x)\,dx \equiv \lim_{a\to -\infty} \int_a^b f(x)\,dx$$

with the same criterion of convergence and divergence.

Example 1 Evaluate $\int_1^{\infty} \frac{dx}{x^2}$. First we note that

$$\int_1^b \frac{dx}{x^2} = \frac{-1}{x}\bigg]_1^b = \frac{-1}{b} + 1$$

Hence, in line with (13.8), the desired integral is

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(\frac{-1}{b} + 1\right) = 1$$

This improper integral does converge, and it has a value of 1.

Since the limit expression is cumbersome to write, some people prefer to omit the "lim" notation and write simply

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{-1}{x}\bigg]_1^{\infty} = 0 + 1 = 1$$

Even when written in this form, however, the improper integral should nevertheless be interpreted with the limit concept in mind.

Graphically, this improper integral still has the connotation of an area. But since the upper limit of integration is allowed to take on increasingly larger values in this case, the right-side boundary must be extended eastward indefinitely, as

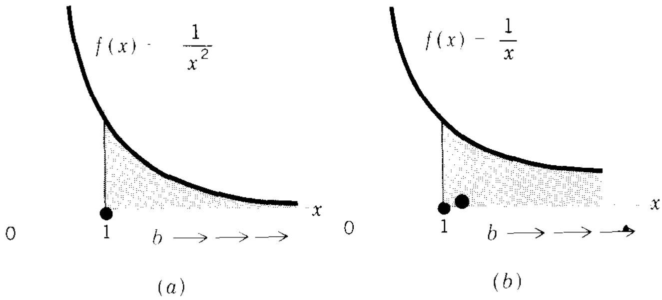

Figure 13.4

shown in Fig. 13.4*a*. Despite this, we are able to consider the area to have the definite (limit) value of 1.

Example 2 Evaluate $\int_1^\infty \frac{dx}{x}$. As before, we first find

$$\int_1^b \frac{dx}{x} = \ln x \Big]_1^b = \ln b - \ln 1 = \ln b$$

When we let $b \to \infty$, by (10.16′) we have $\ln b \to \infty$. Thus the given improper integral is divergent.

Figure 13.4*b* shows the graph of the function $1/x$, as well as the area corresponding to the given integral. The indefinite eastward extension of the right-side boundary will result this time in an infinite area, even though the shape of the graph displays a superficial similarity to that of diagram *a*.

What if both limits of integration are infinite? A direct extension of (13.8) and (13.8′) would suggest the definition

$$(13.8'') \qquad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{b \to +\infty \\ a \to -\infty}} \int_a^b f(x)\,dx$$

Again, this improper integral is said to converge if and only if the limit in question exists.

Infinite Integrand

Even with finite limits of integration, an integral can still be improper if the integrand becomes infinite somewhere in the interval of integration $[a, b]$. To evaluate such an integral, we must again rely upon the concept of a limit.

Example 3 Evaluate $\int_0^1 \frac{1}{x}\,dx$. This integral is improper because, as Fig. 13.4*b* shows, the integrand is infinite at the lower limit of integration ($1/x \to \infty$ as

$x \to 0^+$). Therefore we should first find the integral

$$\int_a^1 \frac{1}{x}\,dx = \ln x\Big]_a^1 = \ln 1 - \ln a = -\ln a \qquad [\text{for } a > 0]$$

and then evaluate its limit as $a \to 0^+$:

$$\int_0^1 \frac{1}{x}\,dx \equiv \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} (-\ln a)$$

Since this limit does not exist (as $a \to 0^+$, $\ln a \to -\infty$), the given integral is divergent.

Example 4 Evaluate $\int_0^9 x^{-1/2}\,dx$. When $x \to 0^+$, the integrand $1/\sqrt{x}$ becomes infinite; the integral is improper. Again, we can first find

$$\int_a^9 x^{-1/2}\,dx = 2x^{1/2}\Big]_a^9 = 6 - 2\sqrt{a}$$

The limit of this expression as $a \to 0^+$ is $6 - 0 = 6$. Thus the given integral is convergent (to 6).

The situation where the integrand becomes infinite at the *upper* limit of integration is perfectly similar. It is an altogether different proposition, however, when an infinite value of the integrand occurs in the open interval (a, b) rather than at a or b. In this eventuality, it is necessary to take advantage of the additivity of definite integrals and first decompose the given integral into subintegrals. Assume that $f(x) \to \infty$ as $x \to p$, where p is a point in the interval (a, b); then, by the additivity property, we have

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$

The given integral on the left can be considered as convergent if and only if each subintegral has a limit.

Example 5 Evaluate $\int_{-1}^1 \frac{1}{x^3}\,dx$. The integrand tends to infinity when x approaches zero; thus we must write the given integral as the sum

$$\int_{-1}^1 x^{-3}\,dx = \int_{-1}^0 x^{-3}\,dx + \int_0^1 x^{-3}\,dx \qquad (\text{say}, \equiv I_1 + I_2)$$

The integral I_1 is divergent, because

$$\lim_{b\to 0^-} \int_{-1}^b x^{-3}\,dx = \lim_{b\to 0^-} \left[\frac{-1}{2}x^{-2}\right]_{-1}^b = \lim_{b\to 0} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = -\infty$$

Thus, we can conclude immediately, without having to evaluate I_2, that the given integral is divergent.

EXERCISE 13.4

1 Check the definite integrals given in Exercises 13.3-1 and 13.3-2 to determine whether any of them is improper. If improper, indicate which variety of improper integral each one is.

2 Which of the following integrals are improper, and why?

(*a*) $\int_0^\infty e^{-rt}\,dt$ (*c*) $\int_0^1 x^{-2/3}\,dx$ (*e*) $\int_1^5 \frac{dx}{x-2}$

(*b*) $\int_2^3 x^4\,dx$ (*d*) $\int_{-\infty}^0 e^{rt}\,dt$ (*f*) $\int_{-3}^4 6\,dx$

3 Evaluate all the *improper* integrals in the preceding problem.

4 Evaluate the integral I_2 of Example 5, and show that it is also divergent.

5 (*a*) Graph the function $y = ce^{-t}$ for nonnegative t, $(c > 0)$, and shade the area under the curve.

(*b*) Write a mathematical expression for this area, and determine whether it is a finite area.

13.5 SOME ECONOMIC APPLICATIONS OF INTEGRALS

Integrals are used in economic analysis in various ways. We shall illustrate a few simple applications in the present section and then show the application to the Domar growth model in the next.

From a Marginal Function to a Total Function

Given a total function (e.g., a total-cost function), the process of differentiation can yield the marginal function (e.g., the marginal-cost function). Because the process of integration is the opposite of differentiation, it should enable us, conversely, to infer the total function from a given marginal function.

Example 1 If the marginal cost (MC) of a firm is the following function of output, $C'(Q) = 2e^{0.2Q}$, and if the fixed cost is $C_F = 90$, find the total-cost function $C(Q)$. By integrating $C'(Q)$ with respect to Q, we find that

$$\int 2e^{0.2Q}\,dQ = 2\frac{1}{0.2}e^{0.2Q} + c = 10e^{0.2Q} + c \tag{13.9}$$

This result may be taken as the desired $C(Q)$ function except that, in view of the arbitrary constant c, the answer appears indeterminate. Fortunately, the information that $C_F = 90$ can be used as an initial condition to definitize the constant. When $Q = 0$, total cost C will consist solely of C_F. Setting $Q = 0$ in the result of (13.9), therefore, we should get a value of 90; that is, $10e^0 + c = 90$. But this would imply that $c = 90 - 10 = 80$. Hence, the total-cost function is

$$C(Q) = 10e^{0.2Q} + 80$$

Note that, unlike the case of (13.2), where the arbitrary constant c has the same value as the initial value of the variable $H(0)$, in the present example we have $c = 80$ but $C(0) \equiv C_F = 90$, so that the two take different values. In general, it should *not* be assumed that the arbitrary constant c will always be equal to the initial value of the total function.

Example 2 If the marginal propensity to save (MPS) is the following function of income, $S'(Y) = 0.3 - 0.1Y^{-1/2}$, and if the aggregate savings S is nil when income Y is 81, find the saving function $S(Y)$. As the MPS is the derivative of the S function, the problem now calls for the integration of $S'(Y)$:

$$S(Y) = \int (0.3 - 0.1Y^{-1/2})\, dY = 0.3Y - 0.2Y^{1/2} + c$$

The specific value of the constant c can be found from the fact that $S = 0$ when $Y = 81$. Even though, strictly speaking, this is not an *initial* condition (not relating to $Y = 0$), substitution of this information into the above integral will nevertheless serve to definitize c. Since

$$0 = 0.3(81) - 0.2(9) + c \quad \Rightarrow \quad c = -22.5$$

the desired saving function is

$$S(Y) = 0.3Y - 0.2Y^{1/2} - 22.5$$

The technique illustrated in the above two examples can be extended directly to other problems involving the search for total functions (such as total revenue, total consumption) from given marginal functions. It may also be reiterated that in problems of this type the validity of the answer (an integral) can always be checked by differentiation.

Investment and Capital Formation

Capital formation is the process of adding to a given stock of capital. Regarding this process as continuous over time, we may express capital stock as a function of time, $K(t)$, and use the derivative dK/dt to denote the rate of capital formation.* But the rate of capital formation at time t is identical with the rate of *net investment* flow at time t, denoted by $I(t)$. Thus, capital stock K and net investment I are related by the following two equations:

$$\frac{dK}{dt} \equiv I(t)$$

and $$K(t) = \int I(t)\, dt = \int \frac{dK}{dt}\, dt = \int dK$$

* As a matter of notation, the derivative of a variable with respect to *time* often is also denoted by a dot placed over the variable, such as $\dot{K} \equiv dK/dt$. In dynamic analysis, where derivatives with respect to *time* occur in abundance, this more concise symbol can contribute substantially to notational simplicity. However, a dot, being such a tiny mark, is easily lost sight of or misplaced; thus, great care is required in using this symbol.

The first equation above is an identity; it shows the synonymity between net investment and the increment of capital. Since $I(t)$ is the derivative of $K(t)$, it stands to reason that $K(t)$ is the integral or antiderivative of $I(t)$, as shown in the second equation. The transformation of the integrand in the latter equation is also easy to comprehend: The switch from I to dK/dt is by definition, and the next transformation is by cancellation of two identical differentials, i.e., by the substitution rule.

Sometimes the concept of *gross investment* is used together with that of net investment in a model. Denoting gross investment by I_g and net investment by I, we can relate them to each other by the equation

$$I_g = I + \delta K$$

where δ represents the rate of depreciation of capital and δK, the rate of *replacement investment.*

Example 3 Suppose that the net investment flow is described by the equation $I(t) = 3t^{1/2}$ and that the initial capital stock, at time $t = 0$, is $K(0)$. What is the time path of capital K? By integrating $I(t)$ with respect to t, we obtain

$$K(t) = \int I(t)\,dt = \int 3t^{1/2}\,dt = 2t^{3/2} + c$$

Next, letting $t = 0$ in the leftmost and rightmost expressions, we find $K(0) = c$. Therefore, the time path of K is

$$K(t) = 2t^{3/2} + K(0) \qquad (13.10)$$

Observe the basic similarity between the results in (13.10) and in (13.2″).

The concept of definite integral enters into the picture when one desires to find the amount of capital formation during some interval of time (rather than the time path of K). Since $\int I(t)\,dt = K(t)$, we may write the definite integral

$$\int_a^b I(t)\,dt = K(t)\Big]_a^b = K(b) - K(a)$$

to indicate the total capital accumulation during the time interval $[a, b]$. Of course, this also represents an area under the $I(t)$ curve. It should be noted, however, that in the graph of the $K(t)$ function, this definite integral would appear instead as a vertical distance—more specifically, as the difference between the two vertical distances $K(b)$ and $K(a)$. (cf. Exercise 13.3-4.)

To appreciate this distinction between $K(t)$ and $I(t)$ more fully, let us emphasize that capital K is a *stock* concept, whereas investment I is a *flow* concept. Accordingly, while $K(t)$ tells us the *amount* of K existing at each point of time, $I(t)$ gives us the information about the *rate* of (net) investment per year (or per period of time) which is prevailing at each point of time. Thus, in order to calculate the *amount* of net investment undertaken (capital accumulation), we must first specify the length of the interval involved. This fact can also be seen when we rewrite the identity $dK/dt \equiv I(t)$ as $dK \equiv I(t)\,dt$, which states that dK, the increment in K, is based not only on $I(t)$, the rate of flow, but also on dt, the time that elapsed. It is this need to specify the time interval in the expression $I(t)\,dt$ that brings the definite integral into the picture, and gives rise to the *area* representation under the $I(t)$—as against the $K(t)$—curve.

Example 4 If net investment is a constant flow at $I(t) = 1000$ (dollars per year), what will be the total net investment (capital formation) during a year, from $t = 0$ to $t = 1$? Obviously, the answer is \$1000; this can be obtained formally as follows:

$$\int_0^1 I(t)\,dt = \int_0^1 1000\,dt = 1000t\Big]_0^1 = 1000$$

You can verify that the same answer will emerge if, instead, the year involved is from $t = 1$ to $t = 2$.

Example 5 If $I(t) = 3t^{1/2}$ (thousands of dollars per year)—a nonconstant flow—what will be the capital formation during the time interval $[1, 4]$, that is, during the second, third, and fourth years? The answer lies in the definite integral

$$\int_1^4 3t^{1/2}\,dt = 2t^{3/2}\Big]_1^4 = 16 - 2 = 14$$

On the basis of the preceding examples, we may express the amount of capital accumulation during the time interval $[0, t]$, for any investment rate $I(t)$, by the definite integral

$$\int_0^t I(t)\,dt = K(t)\Big]_0^t = K(t) - K(0)$$

Figure 13.5 illustrates the case of the time interval $[0, t_0]$. Viewed differently, the

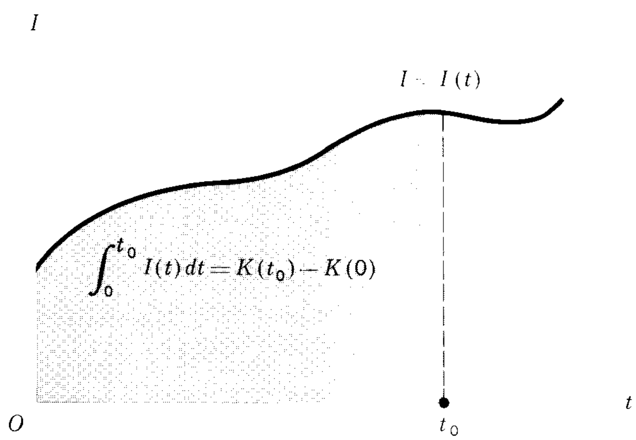

Figure 13.5

above equation yields the following expression for the time path $K(t)$:

$$K(t) = K(0) + \int_0^t I(t)\,dt$$

The amount of K at any time t is the initial capital plus the total capital accumulation that has occurred since.

Present Value of a Cash Flow

Our earlier discussion of discounting and present value, limited to the case of a *single* future value V, led us to the discounting formulas

$$A = V(1+i)^{-t} \qquad [\textit{discrete}\text{ case}]$$

and $$A = Ve^{-rt} \qquad [\textit{continuous}\text{ case}]$$

Now suppose that we have a stream or flow of future values—a series of revenues receivable at various times or of cost outlays payable at various times. How do we compute the present value of the entire cash stream, or cash flow?

In the *discrete* case, if we assume three future revenue figures R_t $(t = 1, 2, 3)$ available at the end of the tth year and also assume an interest rate of i per annum, the present values of R_t will be, respectively,

$$R_1(1+i)^{-1} \qquad R_2(1+i)^{-2} \qquad R_3(1+i)^{-3}$$

It follows that the total present value is the sum

$$\Pi = \sum_{t=1}^{3} R_t(1+i)^{-t} \tag{13.11}$$

(Π is the upper-case Greek letter pi, here signifying *present*.) This differs from the single-value formula only in the replacement of V by R_t and in the insertion of the Σ sign.

The idea of the sum readily carries over to the case of a continuous cash flow, but in the latter context the Σ symbol must give way, of course, to the definite integral sign. Consider a continuous revenue stream at the rate of $R(t)$ dollars per year. This means that at $t = t_1$ the rate of flow is $R(t_1)$ dollars per year, but at another point of time $t = t_2$ the rate will be $R(t_2)$ dollars per year—with t taken as a continuous variable. If at any point of time t we allow an infinitesimal time interval dt to pass, the amount of revenue during the interval $[t, t+dt]$ can be written as $R(t)\,dt$ [cf. the previous discussion of $dK \equiv I(t)\,dt$]. When continuously discounted at the rate of r per year, its present value should be $R(t)e^{-rt}\,dt$. If we let our problem be that of finding the total present value of a three-year stream, our answer is to be found in the following definite integral:

$$\Pi = \int_0^3 R(t)e^{-rt}\,dt \tag{13.11'}$$

This expression, the continuous version of the sum in (13.11), differs from the single-value formula only in the replacement of V by $R(t)$ and in the appending of the definite integral sign.*

Example 6 What is the present value of a continuous revenue flow lasting for y years at the constant rate of D dollars per year and discounted at the rate of r per year? According to (13.11′), we have

$$(13.12) \qquad \Pi = \int_0^y De^{-rt}\,dt = D\int_0^y e^{-rt}\,dt = D\left[\frac{-1}{r}e^{-rt}\right]_0^y$$
$$= \frac{-D}{r}e^{-rt}\Big]_{t=0}^{t=y} = \frac{-D}{r}(e^{-ry} - 1) = \frac{D}{r}(1 - e^{-ry})$$

Thus, Π depends on D, r and y. If $D = \$3000$, $r = 0.06$, and $y = 2$, for instance, we have

$$\Pi = \frac{3000}{0.06}(1 - e^{-0.12}) = 50{,}000(1 - 0.8869) = \$5655 \qquad \text{[approximately]}$$

The value of Π naturally is always positive; this follows from the positivity of D and r, as well as $(1 - e^{-ry})$. (The number e raised to any negative power will always give a positive fractional value, as can be seen from the second quadrant of Fig. 10.3*a*.)

Example 7 In the wine-storage problem of Sec. 10.6, we assumed zero storage cost. That simplifying assumption was necessitated by our ignorance of a way to compute the present value of a cost flow. With this ignorance behind us, we are now ready to permit the wine dealer to incur storage costs.

Let the purchase cost of the case of wine be an amount C, incurred at the present time. Its (future) sale value, which varies with time, may be generally denoted as $V(t)$—its present value being $V(t)e^{-rt}$. Whereas the sale value represents a single future value (there can be only one sale transaction on this case of wine), the storage cost is a stream. Assuming this cost to be a constant stream at the rate of s dollars per year, the total present value of the storage cost incurred in a total of t years will amount to

$$\int_0^t se^{-rt}\,dt = \frac{s}{r}(1 - e^{-rt}) \qquad \text{[cf. (13.12)]}$$

Thus the *net* present value—what the dealer would seek to maximize—can be expressed as

$$N(t) = V(t)e^{-rt} - \frac{s}{r}(1 - e^{-rt}) - C = \left[V(t) + \frac{s}{r}\right]e^{-rt} - \frac{s}{r} - C$$

which is an objective function in a single choice variable t.

* It may be noted that, whereas the upper summation index and the upper limit of integration are identical at 3, the lower summation index 1 differs from the lower limit of integration 0. This is because the first revenue in the discrete stream, by assumption, will not be forthcoming until $t = 1$ (end of first year), but the revenue flow in the continuous case is assumed to commence immediately after $t = 0$.