CHAPTER

THREE

EQUILIBRIUM ANALYSIS IN ECONOMICS

The analytical procedure outlined in the preceding chapter will first be applied to what is known as *static analysis*, or *equilibrium analysis*. For this purpose, it is imperative first to have a clear understanding of what "equilibrium" means.

3.1 THE MEANING OF EQUILIBRIUM

Like any economic term, *equilibrium* can be defined in various ways. According to one definition, an equilibrium is "a constellation of selected interrelated variables so adjusted to one another that no inherent tendency to change prevails in the model which they constitute."* Several words in this definition deserve special attention. First, the word "selected" underscores the fact that there do exist variables which, by the analyst's choice, have not been included in the model. Hence the equilibrium under discussion can have relevance only in the context of the particular set of variables chosen, and if the model is enlarged to include additional variables, the equilibrium state pertaining to the smaller model will no longer apply.

Second, the word "interrelated" suggests that, in order for equilibrium to obtain, all variables in the model must simultaneously be in a state of rest. Moreover, the state of rest of each variable must be compatible with that of every

* Fritz Machlup, "Equilibrium and Disequilibrium: Misplaced Concreteness and Disguised Politics," *Economic Journal*, March 1958, p. 9. (Reprinted in F. Machlup, *Essays on Economic Semantics*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.)

other variable; otherwise some variable(s) will be changing, thereby also causing the others to change in a chain reaction, and no equilibrium can be said to exist.

Third, the word "inherent" implies that, in defining an equilibrium, the state of rest involved is based only on the balancing of the internal forces of the model, while the external factors are assumed fixed. Operationally, this means that parameters and exogenous variables are treated as constants. When the external factors do actually change, there may result a new equilibrium defined on the basis of the new parameter values, but in defining the new equilibrium, the new parameter values are again assumed to persist and stay unchanged.

In essence, an equilibrium for a specified model is a situation that is characterized by a lack of tendency to change. It is for this reason that the analysis of equilibrium (more specifically, the study of what the equilibrium state is like) is referred to as *statics*. The fact that an equilibrium implies no tendency to change may tempt one to conclude that an equilibrium necessarily constitutes a desirable or ideal state of affairs, on the ground that only in the ideal state would there be a lack of motivation for change. Such a conclusion is unwarranted. Even though a certain equilibrium position may represent a desirable state and something to be striven for—such as a profit-maximizing situation, from the firm's point of view—another equilibrium position may be quite undesirable and therefore something to be avoided, such as an underemployment equilibrium level of national income. The only warranted interpretation is that an equilibrium is a situation which, if attained, would tend to perpetuate itself, barring any changes in the external forces.

The desirable variety of equilibrium, which we shall refer to as *goal equilibrium*, will be treated later in Parts 4 and 6 as optimization problems. In the present chapter, the discussion will be confined to the *nongoal* type of equilibrium, resulting not from any conscious aiming at a particular objective but from an impersonal or suprapersonal process of interaction and adjustment of economic forces. Examples of this are the equilibrium attained by a market under given demand and supply conditions and the equilibrium of national income under given conditions of consumption and investment patterns.

3.2 PARTIAL MARKET EQUILIBRIUM—A LINEAR MODEL

In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium condition of the model. This is because once we have identified those values, we have in effect identified the equilibrium state. Let us illustrate with a so-called "partial-equilibrium market model," i.e., a model of price determination in an isolated market.

Constructing the Model

Since only one commodity is being considered, it is necessary to include only three variables in the model: the quantity demanded of the commodity (Q_d), the quantity supplied of the commodity (Q_s), and its price (P). The quantity is measured, say, in pounds per week, and the price in dollars. Having chosen the variables, our next order of business is to make certain assumptions regarding the working of the market. First, we must specify an equilibrium condition—something indispensable in an equilibrium model. The standard assumption is that equilibrium obtains in the market if and only if the excess demand is zero ($Q_d - Q_s = 0$), that is, if and only if the market is cleared. But this immediately raises the question of how Q_d and Q_s themselves are determined. To answer this, we assume that Q_d is a decreasing linear function of P (as P increases, Q_d decreases). On the other hand, Q_s is postulated to be an increasing linear function of P (as P increases, so does Q_s), with the proviso that no quantity is supplied unless the price exceeds a particular positive level. In all, then, the model will contain one equilibrium condition plus two behavioral equations which govern the demand and supply sides of the market, respectively.

Translated into mathematical statements, the model can be written as:

$$
\begin{aligned}
&Q_d = Q_s \\
(3.1) \qquad &Q_d = a - bP \qquad (a, b > 0) \\
&Q_s = -c + dP \qquad (c, d > 0)
\end{aligned}
$$

Four parameters, a, b, c, and d, appear in the two linear functions, and all of them are specified to be positive. When the demand function is graphed, as in Fig. 3.1, its vertical intercept is at a and its slope is $-b$, which is negative, as required. The supply function also has the required type of slope, d being positive, but its

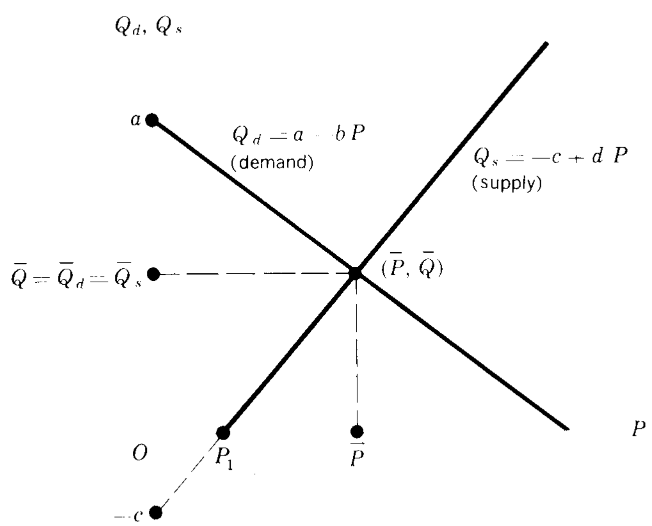

Figure 3.1

vertical intercept is seen to be negative, at $-c$. Why did we want to specify such a negative vertical intercept? The answer is that, in so doing, we force the supply curve to have a positive horizontal intercept at P_1, thereby satisfying the proviso stated earlier that supply will not be forthcoming unless the price is positive and sufficiently high.

The reader should observe that, contrary to the usual practice, quantity rather than price has been plotted vertically in Fig. 3.1. This, however, is in line with the mathematical convention of placing the *dependent* variable on the vertical axis. In a different context below, in which the demand curve is viewed from the standpoint of a business firm as describing the average-revenue curve, $AR \equiv P = f(Q_d)$, we shall reverse the axes and plot P vertically.

With the model thus constructed, the next step is to solve it, i.e., to obtain the solution values of the three endogenous variables, Q_d, Q_s, and P. The solution values, to be denoted $\bar{Q}_d$, $\bar{Q}_s$, and $\bar{P}$, are those values that satisfy the three equations in (3.1) simultaneously; i.e., they are the values which, when substituted into the three equations, make the latter a set of true statements. In the context of an equilibrium model, those values may also be referred to as the *equilibrium values* of the said variables. Since $\bar{Q}_d = \bar{Q}_s$, however, they can be replaced by a single symbol $\bar{Q}$. Hence, an equilibrium solution of the model may simply be denoted by an ordered pair $(\bar{P}, \bar{Q})$. In case the solution is not unique, several ordered pairs may each satisfy the system of simultaneous equations; there will then be a solution set with more than one element in it. However, the multiple-equilibrium situation cannot arise in a linear model such as the present one.

Solution by Elimination of Variables

One way of finding a solution to an equation system is by successive elimination of variables and equations through substitution. In (3.1), the model contains three equations in three variables. However, in view of the equating of Q_d and Q_s by the equilibrium condition, we can let $Q = Q_d = Q_s$ and rewrite the model equivalently as follows:

$$\text{(3.2)} \qquad \begin{aligned} Q &= a - bP \\ Q &= -c + dP \end{aligned}$$

thereby reducing the model to two equations in two variables. Moreover, by substituting the first equation into the second in (3.2), the model can be further reduced to a single equation in a single variable:

$$a - bP = -c + dP$$

or, after subtracting $(a + dP)$ from both sides of the equation and multiplying through by -1,

$$\text{(3.3)} \qquad (b + d)P = a + c$$

This result is also obtainable directly from (3.1) by substituting the second and third equations into the first.

Since $b + d \neq 0$, it is permissible to divide both sides of (3.3) by $(b + d)$. The result is the solution value of P:

$$(3.4) \qquad \bar{P} = \frac{a + c}{b + d}$$

Note that $\bar{P}$ is—as all solution values should be—expressed entirely in terms of the parameters, which represent given data for the model. Thus $\bar{P}$ is a determinate value, as it ought to be. Also note that $\bar{P}$ is positive—as a price should be—because all the four parameters are positive by model specification.

To find the equilibrium quantity $\bar{Q}$ $(= \bar{Q}_d = \bar{Q}_s)$ that corresponds to the value $\bar{P}$, simply substitute (3.4) into *either* equation of (3.2), and then solve the resulting equation. Substituting (3.4) into the demand function, for instance, we can get

$$(3.5) \qquad \bar{Q} = a - \frac{b(a + c)}{b + d} = \frac{a(b + d) - b(a + c)}{b + d} = \frac{ad - bc}{b + d}$$

which is again an expression in terms of parameters only. Since the denominator $(b + d)$ is positive, the positivity of $\bar{Q}$ requires that the numerator $(ad - bc)$ be positive as well. Hence, to be economically meaningful, the present model should contain the additional restriction that $ad > bc$.

The meaning of this restriction can be seen in Fig. 3.1. It is well known that the $\bar{P}$ and $\bar{Q}$ of a market model may be determined graphically at the intersection of the demand and supply curves. To have $\bar{Q} > 0$ is to require the intersection point to be located above the horizontal axis in Fig. 3.1, which in turn requires the slopes and vertical intercepts of the two curves to fulfill a certain restriction on their relative magnitudes. That restriction, according to (3.5), is $ad > bc$, given that both b and d are positive.

The intersection of the demand and supply curves in Fig. 3.1, incidentally, is in concept no different from the intersection shown in the Venn diagram of Fig. 2.2*b*. There is one difference only: instead of the points lying within two circles, the present case involves the points that lie on two lines. Let the set of points on the demand and supply curves be denoted, respectively, by D and S. Then, by utilizing the symbol Q $(= Q_d = Q_s)$, the two sets and their intersection can be written

$$D = \{(P, Q) \mid Q = a - bP\}$$

$$S = \{(P, Q) \mid Q = -c + dP\}$$

and $\qquad D \cap S = (\bar{P}, \bar{Q})$

The intersection set contains in this instance only a single element, the ordered pair $(\bar{P}, \bar{Q})$. The market equilibrium is unique.

EXERCISE 3.2

1 Given the market model

$$Q_d = Q_s$$
$$Q_d = 24 - 2P$$
$$Q_s = -5 + 7P$$

find $\bar{P}$ and $\bar{Q}$ by (*a*) elimination of variables and (*b*) using formulas (3.4) and (3.5). (Use fractions rather than decimals.)

2 Let the demand and supply functions be as follows:

(*a*) $Q_d = 51 - 3P$ $\quad$ (*b*) $Q_d = 30 - 2P$

$Q_s = 6P - 10$ $\quad$ $Q_s = -6 + 5P$

find $\bar{P}$ and $\bar{Q}$ by elimination of variables. (Use fractions rather than decimals.)

3 According to (3.5), for $\bar{Q}$ to be positive, it is necessary that the expression $(ad - bc)$ have the same algebraic sign as $(b + d)$. Verify that this condition is indeed satisfied in the models of the preceding two problems.

4 If $(b + d) = 0$ in the linear market model, can an equilibrium solution be found by using (3.4) and (3.5)? Why or why not?

5 If $(b + d) = 0$ in the linear market model, what can you conclude regarding the positions of the demand and supply curves in Fig. 3.1? What can you conclude, then, regarding the equilibrium solution?

3.3 PARTIAL MARKET EQUILIBRIUM—A NONLINEAR MODEL

Let the linear demand in the isolated market model be replaced by a quadratic demand function, while the supply function remains linear. Then, if numerical coefficients are employed rather than parameters, a model such as the following may emerge:

$$\begin{aligned} Q_d &= Q_s \\ Q_d &= 4 - P^2 \\ Q_s &= 4P - 1 \end{aligned} \tag{3.6}$$

As previously, this system of three equations can be reduced to a single equation by elimination of variables (by substitution):

$$4 - P^2 = 4P - 1$$

or

$$P^2 + 4P - 5 = 0 \tag{3.7}$$

This is a quadratic equation because the left-hand expression is a quadratic function of variable P. The major difference between a quadratic equation and a linear one is that, in general, the former will yield two solution values.

Quadratic Equation versus Quadratic Function

Before discussing the method of solution, a clear distinction should be made between the two terms *quadratic equation* and *quadratic function*. According to the earlier discussion, the expression $P^2 + 4P - 5$ constitutes a quadratic *function*, say, $f(P)$. Hence we may write

(3.8) $$f(P) = P^2 + 4P - 5$$

What (3.8) does is to specify a rule of mapping from P to $f(P)$, such as

P	$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	$\cdots$
$f(P)$	$\cdots$	7	0	-5	-8	-9	-8	-5	0	7	$\cdots$

Although we have listed only nine P values in this table, actually *all* the P values in the domain of the function are eligible for listing. It is perhaps for this reason that we rarely speak of "solving" the equation $f(P) = P^2 + 4P - 5$, because we normally expect "solution values" to be few in number, but here all P values can get involved. Nevertheless, one may legitimately consider each ordered pair in the table above—such as $(-6, 7)$ and $(-5, 0)$—as a solution of (3.8), since each such ordered pair indeed satisfies that equation. Inasmuch as an infinite number of such ordered pairs can be written, one for each P value, there is an infinite number of solutions to (3.8). When plotted as a curve, these ordered pairs together yield the parabola in Fig. 3.2.

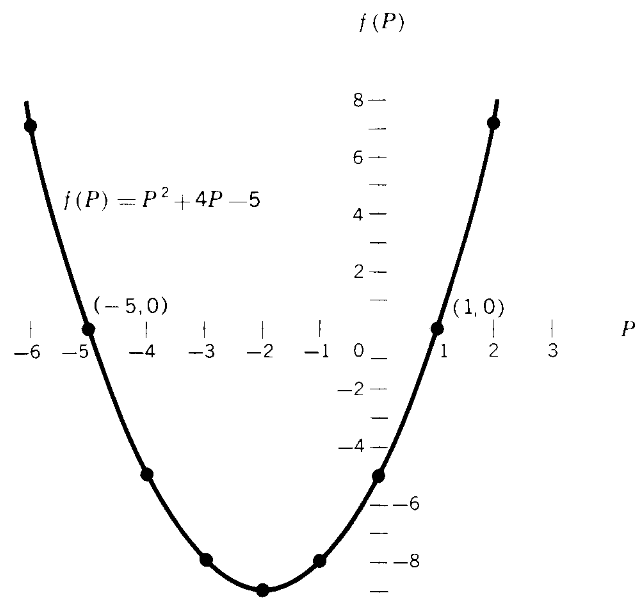

Figure 3.2

In (3.7), where we set the quadratic function $f(P)$ equal to zero, the situation is fundamentally changed. Since the variable $f(P)$ now disappears (having been assigned a zero value), the result is a quadratic *equation* in the single variable P.* Now that $f(P)$ is restricted to a zero value, only a select number of P values can satisfy (3.7) and qualify as its solution values, namely, those P values at which the parabola in Fig. 3.2 intersects the horizontal axis—on which $f(P)$ is zero. Note that this time the solution values are just P values, not ordered pairs. The solution P values are often referred to as the *roots* of the quadratic *equation* $f(P) = 0$, or, alternatively, as the *zeros* of the quadratic *function* $f(P)$.

There are two such intersection points in Fig. 3.2, namely, $(1, 0)$ and $(-5, 0)$. As required, the second element of each of these ordered pairs (the *ordinate* of the corresponding point) shows $f(P) = 0$ in both cases. The first element of each ordered pair (the *abscissa* of the point), on the other hand, gives the solution value of P. Here we get two solutions,

$$\bar{P}_1 = 1 \qquad \text{and} \qquad \bar{P}_2 = -5$$

but only the first is economically admissible, as negative prices are ruled out.

The Quadratic Formula

Equation (3.7) has been solved graphically, but an algebraic method is also available. In general, given a quadratic equation in the form

$$(3.9) \qquad ax^2 + bx + c = 0 \qquad (a \neq 0)$$

its two roots can be obtained from the *quadratic formula*:

$$(3.10) \qquad \bar{x}_1, \bar{x}_2 = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

where the $+$ part of the $\pm$ sign yields $\bar{x}_1$ and the $-$ part yields $\bar{x}_2$.

This widely used formula is derived by means of a process known as "completing the square." First, dividing each term of (3.9) by a results in the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Subtracting c/a from, and adding $b^2/4a^2$ to, both sides of the equation, we get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

* The distinction between quadratic function and quadratic equation just discussed can be extended also to cases of polynomials other than quadratic. Thus, a cubic equation results when a cubic function is set equal to zero.

The left side is now a "perfect square," and thus the equation can be expressed as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

or, after taking the square root on both sides,

$$x + \frac{b}{2a} = \pm\frac{(b^2 - 4ac)^{1/2}}{2a}$$

Finally, by subtracting $b/2a$ from both sides, the result in (3.10) is evolved.

Applying the formula to (3.7), where $a = 1$, $b = 4$, $c = -5$, and $x = P$, the roots are found to be

$$\bar{P}_1, \bar{P}_2 = \frac{-4 \pm (16 + 20)^{1/2}}{2} = \frac{-4 \pm 6}{2} = 1, -5$$

which check with the graphical solutions in Fig. 3.2. Again, we reject $\bar{P}_2 = -5$ on economic grounds and, after omitting the subscript 1, write simply $\bar{P} = 1$.

With this information in hand, the equilibrium quantity $\bar{Q}$ can readily be found from either the second or the third equation of (3.6) to be $\bar{Q} = 3$.

Another Graphical Solution

One method of graphical solution of the present model has been presented in Fig. 3.2. However, since the quantity variable has been eliminated in deriving the quadratic equation, only $\bar{P}$ can be found from that figure. If we are interested in

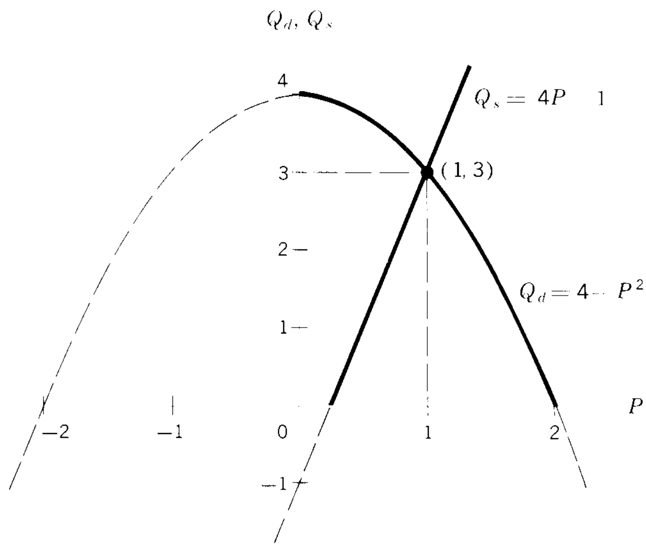

Figure 3.3

finding $\bar{P}$ and $\bar{Q}$ simultaneously from a graph, we must instead use a diagram with Q on one axis and P on the other, similar in construction to Fig. 3.1. This is illustrated in Fig. 3.3. Our problem is of course again to find the intersection of two sets of points, namely,

$$D = \{(P, Q) \mid Q = 4 - P^2\}$$

and $$S = \{(P, Q) \mid Q = 4P - 1\}$$

If no restriction is placed on the domain and the range, the intersection set will contain two elements, namely,

$$D \cap S = \{(1, 3), (-5, -21)\}$$

The former is located in quadrant I, and the latter (not drawn) in quadrant III. If the domain and range are restricted to being nonnegative, however, only the first ordered pair $(1, 3)$ can be accepted. Then the equilibrium is again unique.

Higher-Degree Polynomial Equations

If a system of simultaneous equations reduces not to a linear equation such as (3.3)* or to a quadratic equation such as (3.7) but to a cubic (third-degree polynomial) equation or quartic (fourth-degree polynomial) equation, the roots will be more difficult to find. One useful method which may work is that of *factoring* the function. For example, the expression $x^3 - x^2 - 4x + 4$ can be written as the product of three factors $(x - 1)$, $(x + 2)$, and $(x - 2)$. Thus the cubic equation

$$x^3 - x^2 - 4x + 4 = 0$$

can be written after factoring as

$$(x - 1)(x + 2)(x - 2) = 0$$

In order for the left-hand product to be zero, at least one of the three terms in the product must be zero. Setting each term equal to zero in turn, we get

$$x - 1 = 0 \qquad \text{or} \qquad x + 2 = 0 \qquad \text{or} \qquad x - 2 = 0$$

These three equations will supply the three roots of the cubic equation, namely,

$$\bar{x}_1 = 1 \qquad \bar{x}_2 = -2 \qquad \text{and} \qquad \bar{x}_3 = 2$$

The trick is, of course, to discover the appropriate way of factoring. Unfortunately, no general rule exists, and it must therefore remain a matter of trial and error. Generally speaking, however, given an nth-degree polynomial equation $f(x) = 0$, we can expect exactly n roots, which may be found as follows. First, try to find a constant c_1 such that $f(x)$ is divisible by $(x + c_1)$. The quotient $f(x)/(x + c_1)$ will be a polynomial function of a lesser—$(n - 1)$st—degree; let

* Equation (3.3) can be viewed as the result of setting the linear function $(b + d)P - (a + c)$ equal to zero.

us call it $g(x)$. It then follows that

$$f(x) = (x + c_1)g(x)$$

Now, try to find a constant c_2 such that $g(x)$ is divisible by $(x + c_2)$. The quotient $g(x)/(x + c_2)$ will again be a polynomial function of a lesser—this time $(n - 2)$nd—degree, say, $h(x)$. Since $g(x) = (x + c_2)h(x)$, it follows that

$$f(x) = (x + c_1)g(x) = (x + c_1)(x + c_2)h(x)$$

By repeating the process, it will be possible to reduce the original nth-degree polynomial $f(x)$ to a product of exactly n terms:

$$f(x) = (x + c_1)(x + c_2) \cdots (x + c_n)$$

which, when set equal to zero, will yield n roots. Setting the first factor equal to zero, for example, one gets $\bar{x}_1 = -c_1$. Similarly, the other factors will yield $\bar{x}_2 = -c_2$, $\bar{x}_3 = -c_3$, etc. These results can be more succinctly expressed by employing an *index subscript* i:

$$\bar{x}_i = -c_i \qquad (i = 1, 2, \ldots, n)$$

Even though only one equation is written, the fact that the subscript i can take n different values means that in all there are n equations involved. Thus the index subscript provides a very concise way of statement.

EXERCISE 3.3

1 Find the zeros of the following functions graphically:
(a) $f(x) = x^2 - 7x + 10$ (b) $g(x) = 2x^2 - 4x - 16$

2 Solve the preceding problem by the quadratic formula.

3 Solve the following polynomial equations by factoring:
(a) $P^2 + 4P - 5 = 0$ [see (3.7)] (c) $x^3 - 7x^2 + 14x - 8 = 0$
(b) $x^3 + 2x^2 - 4x - 8 = 0$ (d) $x^3 - 3x^2 - 4x = 0$

4 Find a cubic function with roots 7, -2, and 5.

5 Find the equilibrium solution for each of the following models:
(a) $Q_d = Q_s$ (b) $Q_d = Q_s$
$Q_d = 3 - P^2$ $Q_d = 8 - P^2$
$Q_s = 6P - 4$ $Q_s = P^2 - 2$

6 The market equilibrium condition, $Q_d = Q_s$, is often expressed in an equivalent alternative form, $Q_d - Q_s = 0$, which has the economic interpretation "excess demand is zero." Does (3.7) represent this latter version of the equilibrium condition? If not, supply an appropriate economic interpretation for (3.7).

3.4 GENERAL MARKET EQUILIBRIUM

The last two sections dealt with models of an isolated market, wherein the Q_d and Q_s of a commodity are functions of the price of that commodity alone. In the actual world, though, no commodity ever enjoys (or suffers) such a hermitic existence; for every commodity, there would normally exist many substitutes and complementary goods. Thus a more realistic depiction of the demand function of a commodity should take into account the effect not only of the price of the commodity itself but also of the prices of most, if not all, of the related commodities. The same also holds true for the supply function. Once the prices of other commodities are brought into the picture, however, the structure of the model itself must be broadened so as to be able to yield the equilibrium values of these other prices as well. As a result, the price and quantity variables of multiple commodities must enter endogenously into the model en masse.

In an isolated-market model, the equilibrium condition consists of only one equation, $Q_d = Q_s$, or $E \equiv Q_d - Q_s = 0$, where E stands for excess demand. When several interdependent commodities are simultaneously considered, equilibrium would require the absence of excess demand for each and every commodity included in the model, for if so much as *one* commodity is faced with an excess demand, the price adjustment of that commodity will necessarily affect the quantities demanded and quantities supplied of the remaining commodities, thereby causing price changes all around. Consequently, the equilibrium condition of an n-commodity market model will involve n equations, one for each commodity, in the form

$$(3.11) \qquad E_i \equiv Q_{di} - Q_{si} = 0 \qquad (i = 1, 2, \ldots, n)$$

If a solution exists, there will be a set of prices $\bar{P}_i$ and corresponding quantities $\bar{Q}_i$ such that all the n equations in the equilibrium condition will be simultaneously satisfied.

Two-Commodity Market Model

To illustrate the problem, let us discuss a simple model in which only two commodities are related to each other. For simplicity, the demand and supply functions of both commodities are assumed to be linear. In parametric terms, such a model can be written as

$$(3.12) \qquad \begin{aligned} Q_{d1} - Q_{s1} &= 0 \\ Q_{d1} &= a_0 + a_1 P_1 + a_2 P_2 \\ Q_{s1} &= b_0 + b_1 P_1 + b_2 P_2 \\ Q_{d2} - Q_{s2} &= 0 \\ Q_{d2} &= \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \\ Q_{s2} &= \beta_0 + \beta_1 P_1 + \beta_2 P_2 \end{aligned}$$

where the a and b coefficients pertain to the demand and supply functions of the first commodity, and the α and β coefficients are assigned to those of the second. We have not bothered to specify the signs of the coefficients, but in the course of analysis certain restrictions will emerge as a prerequisite to economically sensible results. Also, in a subsequent numerical example, some comments will be made on the specific signs to be given the coefficients.

As a first step toward the solution of this model, we can again resort to elimination of variables. By substituting the second and third equations into the first (for the first commodity) and the fifth and sixth equations into the fourth (for the second commodity), the model is reduced to two equations in two variables:

$$(3.13)\qquad \begin{aligned} (a_0 - b_0) + (a_1 - b_1)P_1 + (a_2 - b_2)P_2 &= 0 \\ (\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)P_1 + (\alpha_2 - \beta_2)P_2 &= 0 \end{aligned}$$

These represent the two-commodity version of (3.11), after the demand and supply functions have been substituted into the two equilibrium-condition equations.

Although this is a simple system of only two equations, as many as 12 parameters are involved, and algebraic manipulations will prove unwieldy unless some sort of shorthand is introduced. Let us therefore define the shorthand symbols

$$\begin{aligned} c_i &\equiv a_i - b_i \\ \gamma_i &\equiv \alpha_i - \beta_i \end{aligned} \qquad (i = 0, 1, 2)$$

Then (3.13) becomes—after transposing the c_0 and γ_0 terms to the right-hand side of the equals sign:

$$(3.13')\qquad \begin{aligned} c_1P_1 + c_2P_2 &= -c_0 \\ \gamma_1P_1 + \gamma_2P_2 &= -\gamma_0 \end{aligned}$$

which may be solved by further elimination of variables. From the first equation, it can be found that $P_2 = -(c_0 + c_1P_1)/c_2$. Substituting this into the second equation and solving, we get

$$(3.14)\qquad \bar{P}_1 = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}$$

Note that $\bar{P}_1$ is entirely expressed, as a solution value should be, in terms of the data (parameters) of the model. By a similar process, the equilibrium price of the second commodity is found to be

$$(3.15)\qquad \bar{P}_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

For these two values to make sense, however, certain restrictions should be imposed on the model. First, since division by zero is undefined, we must require the common denominator of (3.14) and (3.15) to be nonzero, that is, $c_1\gamma_2 \neq c_2\gamma_1$. Second, to assure positivity, the numerator must have the same sign as the denominator.

The equilibrium prices having been found, the equilibrium quantities $\bar{Q}_1$ and $\bar{Q}_2$ can readily be calculated by substituting (3.14) and (3.15) into the second (or third) equation and the fifth (or sixth) equation of (3.12). These solution values will naturally also be expressed in terms of the parameters. (Their actual calculation is left to you as an exercise.)

Numerical Example

Suppose that the demand and supply functions are numerically as follows:

$$\begin{aligned} Q_{d1} &= 10 - 2P_1 + P_2 \\ Q_{s1} &= -2 + 3P_1 \\ Q_{d2} &= 15 + P_1 - P_2 \\ Q_{s2} &= -1 + 2P_2 \end{aligned} \tag{3.16}$$

What will be the equilibrium solution?

Before answering the question, let us take a look at the numerical coefficients. For each commodity, Q_{si} is seen to depend on P_i alone, but Q_{di} is shown as a function of both prices. Note that while P_1 has a negative coefficient in Q_{d1}, as we would expect, the coefficient of P_2 is positive. The fact that a rise in P_2 tends to raise Q_{d1} suggests that the two commodities are substitutes for each other. The role of P_1 in the Q_{d2} function has a similar interpretation.

With these coefficients, the shorthand symbols c_i and γ_i will take the following values:

$$c_0 = 10 - (-2) = 12 \qquad c_1 = -2 - 3 = -5 \qquad c_2 = 1 - 0 = 1$$

$$\gamma_0 = 15 - (-1) = 16 \qquad \gamma_1 = 1 - 0 = 1 \qquad \gamma_2 = -1 - 2 = -3$$

By direct substitution of these into (3.14) and (3.15), we obtain

$$\bar{P}_1 = \tfrac{52}{14} = 3\tfrac{5}{7} \qquad \text{and} \qquad \bar{P}_2 = \tfrac{92}{14} = 6\tfrac{4}{7}$$

And the further substitution of $\bar{P}_1$ and $\bar{P}_2$ into (3.16) will yield

$$\bar{Q}_1 = \tfrac{64}{7} = 9\tfrac{1}{7} \qquad \text{and} \qquad \bar{Q}_2 = \tfrac{85}{7} = 12\tfrac{1}{7}$$

Thus all the equilibrium values turn out positive, as required. In order to preserve the exact values of $\bar{P}_1$ and $\bar{P}_2$ to be used in the further calculation of $\bar{Q}_1$ and $\bar{Q}_2$, it is advisable to express them as fractions rather than decimals.

Could we have obtained the equilibrium prices graphically? The answer is yes. From (3.13), it is clear that a two-commodity model can be summarized by two equations in two variables P_1 and P_2. With known numerical coefficients, both equations can be plotted in the P_1P_2 coordinate plane, and the intersection of the two curves will then pinpoint $\bar{P}_1$ and $\bar{P}_2$.

n-Commodity Case

The above discussion of the multicommodity market has been limited to the case of two commodities, but it should be apparent that we are already moving from *partial-equilibrium* analysis in the direction of *general-equilibrium* analysis. As more commodities enter into a model, there will be more variables and more equations, and the equations will get longer and more complicated. If all the commodities in an economy are included in a comprehensive market model, the result will be a Walrasian type of general-equilibrium model, in which the excess demand for every commodity is considered to be a function of the prices of all the commodities in the economy.

Some of the prices may, of course, carry zero coefficients when they play no role in the determination of the excess demand of a particular commodity; e.g., in the excess-demand function of pianos the price of popcorn may well have a zero coefficient. In general, however, with n commodities in all, we may express the demand and supply functions as follows (using Q_{di} and Q_{si} as function symbols in place of f and g):

$$(3.17) \qquad \begin{aligned} Q_{di} &= Q_{di}(P_1, P_2, \ldots, P_n) \\ Q_{si} &= Q_{si}(P_1, P_2, \ldots, P_n) \end{aligned} \qquad (i = 1, 2, \ldots, n)$$

In view of the index subscript, these two equations represent the totality of the $2n$ functions which the model contains. (These functions are not necessarily linear.) Moreover, the equilibrium condition is itself composed of a set of n equations,

$$(3.18) \qquad Q_{di} - Q_{si} = 0 \qquad (i = 1, 2, \ldots, n)$$

When (3.18) is added to (3.17), the model becomes complete. You should therefore count a total of $3n$ equations.

Upon substitution of (3.17) into (3.18), however, the model can be reduced to a set of n simultaneous equations only:

$$Q_{di}(P_1, P_2, \ldots, P_n) - Q_{si}(P_1, P_2, \ldots, P_n) = 0 \qquad (i = 1, 2, \ldots, n)$$

Besides, inasmuch as $E_i \equiv Q_{di} - Q_{si}$, where E_i is necessarily also a function of all the n prices, the above set of equations may be written alternatively as

$$E_i(P_1, P_2, \ldots, P_n) = 0 \qquad (i = 1, 2, \ldots, n)$$

Solved simultaneously, these n equations will determine the n equilibrium prices $\overline{P}_i$—if a solution does indeed exist. And then the $\overline{Q}_i$ may be derived from the demand or supply functions.

Solution of a General-Equation System

If a model comes equipped with numerical coefficients, as in (3.16), the equilibrium values of the variables will be in numerical terms, too. On a more general level, if a model is expressed in terms of parametric constants, as in (3.12), the equilibrium values will also involve parameters and will hence appear as "formulas," as exemplified by (3.14) and (3.15). If, for greater generality, even the function forms are left unspecified in a model, however, as in (3.17), the manner of expressing the solution values will of necessity be exceedingly general as well.

Drawing upon our experience in parametric models, we know that a solution value is always an expression in terms of the parameters. For a general-function model containing, say, a total of m parameters $(a_1, a_2, \ldots, a_m)$—where m is not necessarily equal to n—the n equilibrium prices can therefore be expected to take the general analytical form of

$$\bar{P}_i = \bar{P}_i(a_1, a_2, \ldots, a_m) \qquad (i = 1, 2, \ldots, n) \tag{3.19}$$

This is a symbolic statement to the effect that the solution value of *each* variable (here, price) is a function of the set of all parameters of the model. As this is a very general statement, it really does not give much detailed information about the solution. But in the general analytical treatment of some types of problem, even this seemingly uninformative way of expressing a solution will prove of use, as will be seen in a later chapter.

Writing such a solution is an easy task. But an important catch exists: the expression in (3.19) can be justified if and only if a *unique* solution does indeed exist, for then and only then can we map the ordered m-tuple $(a_1, a_2, \ldots, a_m)$ into a determinate value for each price $\bar{P}_i$. Yet, unfortunately for us, there is no a priori reason to presume that every model will automatically yield a unique solution. In this connection, it needs to be emphasized that the process of "counting equations and unknowns" does not suffice as a test. Some very simple examples should convince us that an equal number of equations and unknowns (endogenous variables) does not necessarily guarantee the existence of a unique solution.

Consider the three simultaneous-equation systems

$$\begin{aligned} x + y &= 8 \\ x + y &= 9 \end{aligned} \tag{3.20}$$

$$\begin{aligned} 2x + y &= 12 \\ 4x + 2y &= 24 \end{aligned} \tag{3.21}$$

$$\begin{aligned} 2x + 3y &= 58 \\ y &= 18 \\ x + y &= 20 \end{aligned} \tag{3.22}$$

In (3.20), despite the fact that two unknowns are linked together by exactly two equations, there is nevertheless no solution. These two equations happen to be *inconsistent*, for if the sum of x and y is 8, it cannot possibly be 9 at the same time. In (3.21), another case of two equations in two variables, the two equations are *functionally dependent*, which means that one can be derived from (and is implied by) the other. (Here, the second equation is equal to two times the first equation). Consequently, one equation is redundant and may be dropped from the system, leaving in effect only one equation in two unknowns. The solution will then be the equation $y = 12 - 2x$, which yields not a unique ordered pair $(\bar{x}, \bar{y})$ but an infinite number of them, including (0, 12), (1, 10), (2, 8), etc., all of which satisfy that equation. Lastly, the case of (3.22) involves more equations than unknowns, yet the ordered pair (2, 18) does constitute the unique solution to it. The reason is that, in view of the existence of functional dependence among the equations (the first is equal to the second plus twice the third), we have in effect only two independent, consistent equations in two variables.

These simple examples should suffice to convey the importance of *consistency* and *functional independence* as the two prerequisites for application of the process of counting equations and unknowns. In general, in order to apply that process, make sure that (1) the satisfaction of any one equation in the model will not preclude the satisfaction of another and (2) no equation is redundant. In (3.17), for example, the n demand and n supply functions may safely be assumed to be independent of one another, each being derived from a different source—each demand from the decisions of a group of consumers, and each supply from the decisions of a group of firms. Thus each function serves to describe one facet of the market situation, and none is redundant. Mutual consistency may perhaps also be assumed. In addition, the equilibrium-condition equations in (3.18) are also independent and presumably consistent. Therefore the analytical solution as written in (3.19) can in general be considered justifiable.*

For simultaneous-equation models, there exist systematic methods of testing the existence of a unique (or determinate) solution. These would involve, for linear models, an application of the concept of *determinants*, to be introduced in Chap. 5. In the case of nonlinear models, such a test would also require a knowledge of so-called "partial derivatives" and a special type of determinant called the *Jacobian determinant*, which will be discussed in Chaps. 7 and 8.

EXERCISE 3.4

1 Work out the step-by-step solution of (3.13′), thereby verifying the results in (3.14) and (3.15).

2 Rewrite (3.14) and (3.15) in terms of the original parameters of the model in (3.12).

3 The demand and supply functions of a two-commodity market model are as follows:

$$Q_{d1} = 18 - 3P_1 + P_2 \qquad Q_{d2} = 12 + P_1 - 2P_2$$
$$Q_{s1} = -2 + 4P_1 \qquad Q_{s2} = -2 \qquad + 3P_2$$

Find $\bar{P}_i$ and $\bar{Q}_i$ $(i = 1, 2)$. (Use fractions rather than decimals.)

* This is essentially the way that Léon Walras approached the problem of the existence of a general market equilibrium. In the modern literature, there can be found a number of sophisticated mathematical proofs of the existence of a competitive market equilibrium under certain postulated economic conditions. But the mathematics used is advanced. The easiest one to understand is perhaps the proof given in Robert Dorfman, Paul A. Samuelson, and Robert M. Solow, *Linear Programming and Economic Analysis*, McGraw-Hill Book Company, New York, 1958, chapter 13, which you should read *after* having studied Part 6 of the present volume.