(3.24) and (3.25) must be positive. Since the exogenous expenditures I_0 and G_0 are normally positive, as is the parameter a (the vertical intercept of the consumption function), the sign of the numerator expressions will work out, too.

As a check on our calculation, we can add the $\bar{C}$ expression in (3.25) to $(I_0 + G_0)$ and see whether the sum is equal to the $\bar{Y}$ expression in (3.24). If so, the $\bar{C}$ and $\bar{Y}$ values do satisfy the equilibrium condition, and the solution is valid.

This model is obviously one of extreme simplicity and crudity, but other models of national-income determination, in varying degrees of complexity and sophistication, can be constructed as well. In each case, however, the principles involved in the construction and analysis of the model are identical with those already discussed. For this reason, we shall not go into further illustrations here. A more comprehensive national-income model, involving the simultaneous equilibrium of the money market and the goods market, will be discussed in Sec. 8.6 below.

EXERCISE 3.5

1 Given the following model:

$Y = C + I_0 + G_0$

$C = a + b(Y - T) \qquad (a > 0, \quad 0 < b < 1) \qquad [T\text{: taxes}]$

$T = d + tY \qquad (d > 0, \quad 0 < t < 1) \qquad [t\text{: income tax rate}]$

(*a*) How many endogenous variables are there?

(*b*) Find $\bar{Y}$, $\bar{T}$, and $\bar{C}$.

2 Let the national-income model be:

$Y = C + I_0 + G$

$C = a + b(Y - T_0) \qquad (a > 0, \quad 0 < b < 1)$

$G = gY \qquad (0 < g < 1)$

(*a*) Identify the endogenous variables.

(*b*) Give the economic meaning of the parameter g.

(*c*) Find the equilibrium national income.

(*d*) What restriction on the parameters is needed for a solution to exist?

3 Find $\bar{Y}$ and $\bar{C}$ from the following:

$Y = C + I_0 + G_0$

$C = 25 + 6Y^{1/2}$

$I_0 = 16$

$G_0 = 14$

CHAPTER FOUR

LINEAR MODELS AND MATRIX ALGEBRA

For the one-commodity model (3.1), the solutions $\bar{P}$ and $\bar{Q}$ as expressed in (3.4) and (3.5) are relatively simple, even though a number of parameters are involved. As more and more commodities are incorporated into the model, such solution formulas quickly become cumbersome and unwieldy. That was why we had to resort to a little shorthand, even for the two-commodity case—in order that the solutions (3.14) and (3.15) can still be written in a relatively concise fashion. We did not attempt to tackle any three- or four-commodity models, even in the linear version, primarily because we did not yet have at our disposal a method suitable for handling a large system of simultaneous equations. Such a method is found in *matrix algebra*, the subject of this chapter and the next.

Matrix algebra can enable us to do many things. In the first place, it provides a compact way of writing an equation system, even an extremely large one. Second, it leads to a way of testing the existence of a solution by evaluation of a *determinant*—a concept closely related to that of a matrix. Third, it gives a method of finding that solution (if it exists). Since equation systems are encountered not only in static analysis but also in comparative-static and dynamic analyses and in optimization problems, you will find ample application of matrix algebra in almost every chapter that is to follow.

However, one slight "catch" should be mentioned at the outset. Matrix algebra is applicable only to *linear*-equation systems. How realistically linear equations can describe actual economic relationships depends, of course, on the nature of the relationships in question. In many cases, even if some sacrifice of realism is entailed by the assumption of linearity, an assumed linear relationship can produce a sufficiently close approximation to an actual nonlinear relationship to warrant its use. In other cases, the closeness of approximation may also be

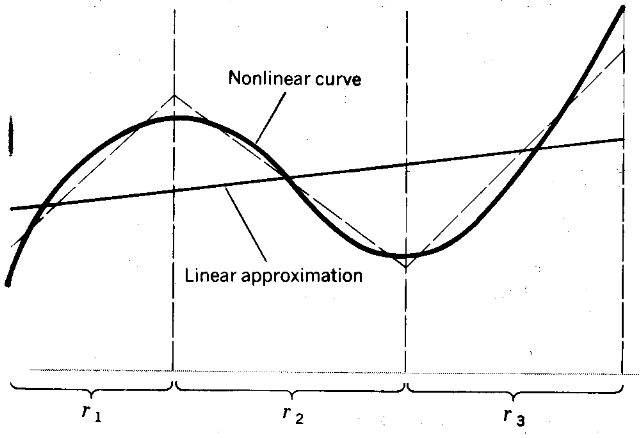

Figure 4.1

improved by having a separate linear approximation for each segment of a nonlinear relationship, as is illustrated in Fig. 4.1. If the solid curve is taken as the actual nonlinear relationship, a single linear approximation might take the form of the solid straight line, which shows substantial deviation from the curve at certain points. But if the domain is divided into three regions r_1, r_2, and r_3, we can have a much closer linear approximation (broken straight line) in each region.

In yet other cases, while preserving the nonlinearity in the model, we can effect a transformation of variables so as to obtain a linear relation to work with. For example, the nonlinear function

$$y = ax^b$$

can be readily transformed, by taking the logarithm on both sides, into the function

$$\log y = \log a + b \log x$$

which is linear in the two variables $(\log y)$ and $(\log x)$. (Logarithms will be discussed in detail in Chap. 10.)

In short, the linearity assumption frequently adopted in economics may in certain cases be quite reasonable and justified. On this note, then, let us proceed to the study of matrix algebra.

4.1 MATRICES AND VECTORS

The two-commodity market model (3.12) can be written—after eliminating the quantity variables—as a system of two linear equations, as in (3.13′),

$$c_1 P_1 + c_2 P_2 = -c_0$$
$$\gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0$$

where the parameters c_0 and γ_0 appear to the right of the equals sign. In general, a system of m linear equations in n variables $(x_1, x_2, \ldots, x_n)$ can also be arranged into such a format:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m
\end{aligned}
\tag{4.1}
$$

In (4.1), the variable x_1 appears only within the leftmost column, and in general the variable x_j appears only in the jth column on the left side of the equals sign. The double-subscripted parameter symbol a_{ij} represents the coefficient appearing in the ith equation and attached to the jth variable. For example, a_{21} is the coefficient in the second equation, attached to the variable x_1. The parameter d_i which is unattached to any variable, on the other hand, represents the constant term in the ith equation. For instance, d_1 is the constant term in the first equation. All subscripts are therefore keyed to the specific locations of the variables and parameters in (4.1).

Matrices as Arrays

There are essentially three types of ingredients in the equation system (4.1). The first is the set of coefficients a_{ij}; the second is the set of variables $x_1, \ldots, x_n$; and the last is the set of constant terms $d_1, \ldots, d_m$. If we arrange the three sets as three rectangular arrays and label them, respectively, as A, x, and d (without subscripts), then we have

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}
\tag{4.2}
$$

As a simple example, given the linear-equation system

$$
\begin{aligned}
6x_1 + 3x_2 + x_3 &= 22 \\
x_1 + 4x_2 - 2x_3 &= 12 \\
4x_1 - x_2 + 5x_3 &= 10
\end{aligned}
\tag{4.3}
$$

we can write

$$
A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}
\tag{4.4}
$$

Each of the three arrays in (4.2) or (4.4) constitutes a *matrix*.

A matrix is defined as a rectangular array of numbers, parameters, or variables. The members of the array, referred to as the *elements* of the matrix, are usually enclosed in brackets, as in (4.2), or sometimes in parentheses or with double vertical lines: || ||. Note that in matrix A (the *coefficient matrix* of the equation system), the elements are separated not by commas but by blank spaces only. As a shorthand device, the array in matrix A can be written more simply as

$$A = [a_{ij}] \qquad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{pmatrix}$$

Inasmuch as the location of each element in a matrix is unequivocally fixed by the subscript, every matrix is an ordered set.

Vectors as Special Matrices

The number of rows and the number of columns in a matrix together define the *dimension* of the matrix. Since matrix A in (4.2) contains m rows and n columns, it is said to be of dimension $m \times n$ (read: "m by n"). It is important to remember that the row number always precedes the column number; this is in line with the way the two subscripts in a_{ij} are ordered. In the special case where $m = n$, the matrix is called a *square matrix*; thus the matrix A in (4.4) is a 3×3 square matrix.

Some matrices may contain only one column, such as x and d in (4.2) or (4.4). Such matrices are given the special name *column vectors*. In (4.2), the dimension of x is $n \times 1$, and that of d is $m \times 1$; in (4.4) both x and d are 3×1. If we arranged the variables x_j in a horizontal array, though, there would result a $1 \times n$ matrix, which is called a *row vector*. For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol:

$$x' = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

You may observe that a vector (whether row or column) is merely an ordered n-tuple, and as such it may be interpreted as a point in an n-dimensional space. In turn, the $m \times n$ matrix A can be interpreted as an ordered set of m row vectors or as an ordered set of n column vectors. These ideas will be followed up later.

An issue of more immediate interest is how the matrix notation can enable us, as promised, to express an equation system in a compact way. With the matrices defined in (4.4), we can express the equation system (4.3) simply as

$$Ax = d$$

In fact, if A, x, and d are given the meanings in (4.2), then even the general-equation system in (4.1) can be written as $Ax = d$. The compactness of this notation is thus unmistakable.

However, the equation $Ax = d$ prompts at least two questions. How do we multiply two matrices A and x? What is meant by the equality of Ax and d? Since matrices involve whole blocks of numbers, the familiar algebraic operations defined for single numbers are not directly applicable, and there is need for a new set of operational rules.

EXERCISE 4.1

1 Rewrite the equation system (3.1) in the format of (4.1), and show that, if the three variables are arranged in the order Q_d, Q_s, and P, the coefficient matrix will be

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$$

How would you write the vector of constants?

2 Rewrite the equation system (3.12) in the format of (4.1) with the variables arranged in the following order: $Q_{d1}, Q_{s1}, Q_{d2}, Q_{s2}, P_1, P_2$. Write out the coefficient matrix, the variable vector, and the constant vector.

4.2 MATRIX OPERATIONS

As a preliminary, let us first define the word *equality*. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if and only if they have the same dimension and have identical elements in the corresponding locations in the array. In other words, $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j. Thus, for example, we find

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

As another example, if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, this will mean that $x = 7$ and $y = 4$.

Addition and Subtraction of Matrices

Two matrices can be added if and only if they have the same dimension. When this dimensional requirement is met, the matrices are said to be *conformable for addition*. In that case, the addition of $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as the addition of each pair of corresponding elements.

Example 1

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$$

Example 2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

In general, we may state the rule thus:

$$[a_{ij}] + [b_{ij}] = [c_{ij}] \qquad \text{where } c_{ij} = a_{ij} + b_{ij}$$

Note that the sum matrix $[c_{ij}]$ must have the same dimension as the component matrices $[a_{ij}]$ and $[b_{ij}]$.

The subtraction operation $A - B$ can be similarly defined if and only if A and B have the same dimension. The operation entails the result

$$[a_{ij}] - [b_{ij}] = [d_{ij}] \qquad \text{where } d_{ij} = a_{ij} - b_{ij}$$

Example 3

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 19-6 & 3-8 \\ 2-1 & 0-3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}$$

The subtraction operation $A - B$ may be considered alternatively as an addition operation involving a matrix A and another matrix $(-1)B$. This, however, raises the question of what is meant by the multiplication of a matrix by a single number (here, -1).

Scalar Multiplication

To multiply a matrix by a number—or in matrix-algebra terminology, by a *scalar* —is to multiply *every* element of that matrix by the given scalar.

Example 4

$$7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}$$

Example 5

$$\frac{1}{2}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{21} & \frac{1}{2}a_{22} \end{bmatrix}$$

From these examples, the rationale of the name scalar should become clear, for it "scales up (or down)" the matrix by a certain multiple. The scalar can, of course, be a negative number as well.

Example 6

$$-1\begin{bmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -d_1 \\ -a_{21} & -a_{22} & -d_2 \end{bmatrix}$$

Note that if the matrix on the left represents the coefficients *and* the constant terms in the simultaneous equations

$$a_{11}x_1 + a_{12}x_2 = d_1$$
$$a_{21}x_1 + a_{22}x_2 = d_2$$

then multiplication by the scalar -1 will amount to multiplying both sides of both equations by -1, thereby changing the sign of every term in the system.

Multiplication of Matrices

Whereas a scalar can be used to multiply a matrix of any dimension, the multiplication of two matrices is contingent upon the satisfaction of a different dimensional requirement.

Suppose that, given two matrices A and B, we want to find the product AB. The conformability condition for multiplication is that the *column* dimension of A (the "lead" matrix in the expression AB) must be equal to the row dimension of B (the "lag" matrix). For instance, if

$$(4.5) \qquad \underset{(1\times 2)}{A} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \qquad \underset{(2\times 3)}{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

the product AB then *is* defined, since A has *two columns* and B has *two rows*—precisely the same number.* This can be checked at a glance by comparing the *second* number in the dimension indicator for A, which is (1×2), with the *first* number in the dimension indicator for B, (2×3). On the other hand, the reverse product BA is *not* defined in this case, because B (now the lead matrix) has *three* columns while A (the lag matrix) has only *one* row; hence the conformability condition is violated.

In general, if A is of dimension $m \times n$ and B is of dimension $p \times q$, the matrix product AB will be defined if and only if $n = p$. If defined, moreover, the product matrix AB will have the dimension $m \times q$—the same number of *rows* as the lead matrix A and the same number of *columns* as the lag matrix B. For the matrices given in (4.5), AB will be 1×3.

It remains to define the exact procedure of multiplication. For this purpose, let us take the matrices A and B in (4.5) for illustration. Since the product AB is defined and is expected to be of dimension 1×3, we may write in general (using the symbol C rather than c' for the row vector) that

$$AB = C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \end{bmatrix}$$

Each element in the product matrix C, denoted by c_{ij}, is defined as a sum of products, to be computed from the elements in the ith *row* of the lead matrix A, and those in the jth *column* of the lag matrix B. To find c_{11}, for instance, we should take the *first row* in A (since $i = 1$) and the *first column* in B (since $j = 1$)

* The matrix A, being a row vector, would normally be denoted by a'. We use the symbol A here to stress the fact that the multiplication rule being explained applies to matrices in general, not only to the product of one vector and one matrix.

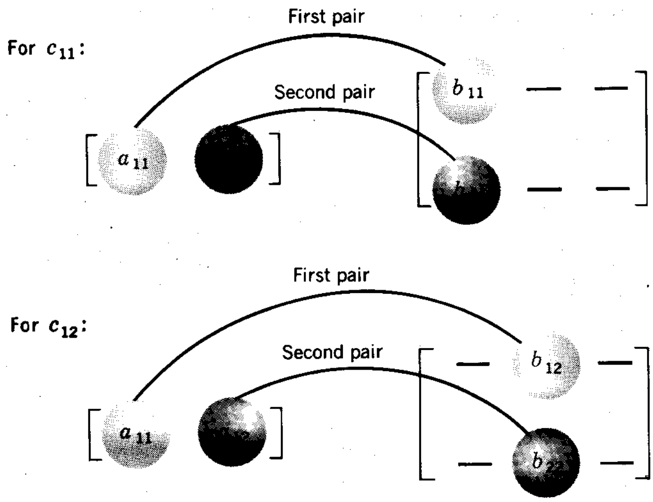

Figure 4.2

—as shown in the top panel of Fig. 4.2—and then pair the elements together sequentially, multiply out each pair, and take the sum of the resulting products, to get

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} \tag{4.6}$$

Similarly, for c_{12}, we take the *first row* in A (since $i = 1$) and the *second column* in B (since $j = 2$), and calculate the indicated sum of products—in accordance with the lower panel of Fig. 4.2—as follows:

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} \tag{4.6'}$$

By the same token, we should also have

$$c_{13} = a_{11}b_{13} + a_{12}b_{23} \tag{4.6''}$$

It is the particular pairing requirement in this process which necessitates the matching of the column dimension of the lead matrix and the row dimension of the lag matrix before multiplication can be performed.

The multiplication procedure illustrated in Fig. 4.2 can also be described by using the concept of the *inner product* of two vectors. Given two vectors u and v with n elements each, say, $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$, arranged *either* as two rows *or* as two columns *or* as one row and one column, their inner product, written as $u \cdot v$, is defined as

$$u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

This is a sum of products of corresponding elements, and hence the inner product of two vectors is a scalar. If, for instance, we prepare after a shopping trip a vector of quantities purchased of n goods and a vector of their prices (listed in the corresponding order), then their inner product will give the total purchase cost.

Note that the inner-product concept is exempted from the conformability condition, since the arrangement of the two vectors in rows or columns is immaterial.

Using this concept, we can describe the element c_{ij} in the product matrix $C = AB$ simply as the inner product of the ith row of the lead matrix A and the jth column of the lag matrix B. By examining Fig. 4.2, we can easily verify the validity of this description.

The rule of multiplication outlined above applies with equal validity when the dimensions of A and B are other than those illustrated above; the only prerequisite is that the conformability condition be met.

Example 7 Given

$$\underset{(2\times2)}{A} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad \underset{(2\times2)}{B} = \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$$

find AB. The product AB is obviously defined, and will be 2×2:

$$AB = \begin{bmatrix} 3(-1) + 5(4) & 3(0) + 5(7) \\ 4(-1) + 6(4) & 4(0) + 6(7) \end{bmatrix} = \begin{bmatrix} 17 & 35 \\ 20 & 42 \end{bmatrix}$$

Example 8 Given

$$\underset{(3\times2)}{A} = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \underset{(2\times1)}{b} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

find Ab. This time the product matrix should be 3×1, that is, a column vector:

$$Ab = \begin{bmatrix} 1(5) + 3(9) \\ 2(5) + 8(9) \\ 4(5) + 0(9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Example 9 Given

$$\underset{(3\times3)}{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \underset{(3\times3)}{B} = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{3}{10} \\ -1 & \frac{1}{5} & \frac{7}{10} \\ 0 & \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$$

find AB. The same rule of multiplication now yields a very special product matrix:

$$AB = \begin{bmatrix} 0 + 1 + 0 & -\frac{3}{5} - \frac{1}{5} + \frac{4}{5} & \frac{9}{10} - \frac{7}{10} - \frac{2}{10} \\ 0 + 0 + 0 & -\frac{1}{5} + 0 + \frac{6}{5} & \frac{3}{10} + 0 - \frac{3}{10} \\ 0 + 0 + 0 & -\frac{4}{5} + 0 + \frac{4}{5} & \frac{12}{10} + 0 - \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This last matrix—a square matrix with 1s in its *principal diagonal* (the diagonal running from northwest to southeast) and 0s everywhere else—exemplifies the important type of matrix known as *identity matrix*. This will be further discussed below.

Example 10 Let us now take the matrix A and the vector x as defined in (4.4) and find Ax. The product matrix is a 3×1 column vector:

$$Ax = \underset{(3\times 3)}{\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}} \underset{(3\times 1)}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{(3\times 1)}{\begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 - 2x_3 \\ 4x_1 - x_2 + 5x_3 \end{bmatrix}}$$

Repeat: the product on the right is a *column* vector, its corpulent appearance notwithstanding! When we write $Ax = d$, therefore, we have

$$\begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ x_1 + 4x_2 - 2x_3 \\ 4x_1 - x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

which, according to the definition of matrix equality, is equivalent to the statement of the entire equation system in (4.3).

Note that, to use the matrix notation $Ax = d$, it is necessary, because of the conformability condition, to arrange the variables x_j into a *column* vector, even though these variables are listed in a horizontal order in the original equation system.

Example 11 The simple national-income model in two endogenous variables Y and C,

$$Y = C + I_0 + G_0$$
$$C = a + bY$$

can be rearranged into the standard format of (4.1) as follows:

$$Y - C = I_0 + G_0$$
$$-bY + C = a$$

Hence the coefficient matrix A, the vector of variables x, and the vector of constants d are:

$$\underset{(2\times 2)}{A} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \qquad \underset{(2\times 1)}{x} = \begin{bmatrix} Y \\ C \end{bmatrix} \qquad \underset{(2\times 1)}{d} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

Let us verify that this given system can be expressed by the equation $Ax = d$.

By the rule of matrix multiplication, we have

$$Ax = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 1(Y) + (-1)(C) \\ -b(Y) + 1(C) \end{bmatrix} = \begin{bmatrix} Y - C \\ -bY + C \end{bmatrix}$$

Thus the matrix equation $Ax = d$ would give us

$$\begin{bmatrix} Y - C \\ -bY + C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

Since matrix equality means the equality between corresponding elements, it is clear that the equation $Ax = d$ does precisely represent the original equation system, as expressed in the (4.1) format above.

The Question of Division

While matrices, like numbers, can undergo the operations of addition, subtraction, and multiplication—subject to the conformability conditions—it is not possible to divide one matrix by another. That is, we cannot write A/B.

For two numbers a and b, the quotient a/b (with $b \neq 0$) can be written alternatively as ab^{-1} or $b^{-1}a$, where b^{-1} represents the *inverse* or *reciprocal* of b. Since $ab^{-1} = b^{-1}a$, the quotient expression a/b can be used to represent both ab^{-1} and $b^{-1}a$. The case of matrices is different. Applying the concept of inverses to matrices, we may in certain cases (discussed below) define a matrix B^{-1} that is the inverse of matrix B. But from the discussion of conformability condition it follows that, if AB^{-1} is defined, there can be no assurance that $B^{-1}A$ is also defined. Even if AB^{-1} and $B^{-1}A$ are indeed both defined, they still may not represent the same product. Hence the expression A/B cannot be used without ambiguity, and it must be avoided. Instead, you must specify whether you are referring to AB^{-1} or $B^{-1}A$—provided that the inverse B^{-1} does exist and that the matrix product in question is defined. Inverse matrices will be further discussed below.

Digression on Σ Notation

The use of subscripted symbols not only helps in designating the locations of parameters and variables but also lends itself to a flexible shorthand for denoting sums of terms, such as those which arose during the process of matrix multiplication.

The summation shorthand makes use of the Greek letter Σ (sigma, for "sum"). To express the sum of x_1, x_2, and x_3, for instance, we may write

$$x_1 + x_2 + x_3 = \sum_{j=1}^{3} x_j$$

which is read: "the sum of x_j as j ranges from 1 to 3." The symbol j, called the *summation index*, takes only integer values. The expression x_j represents the *summand* (that which is to be summed), and it is in effect a function of j. Aside from the letter j, summation indices are also commonly denoted by i or k, such as

$$\sum_{i=3}^{7} x_i = x_3 + x_4 + x_5 + x_6 + x_7$$

$$\sum_{k=0}^{n} x_k = x_0 + x_1 + \cdots + x_n$$