4.4 COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS

In ordinary scalar algebra, the additive and multiplicative operations obey the commutative, associative, and distributive laws as follows:

Commutative law of addition:	$a + b = b + a$
Commutative law of multiplication:	$ab = ba$
Associative law of addition:	$(a + b) + c = a + (b + c)$
Associative law of multiplication:	$(ab)c = a(bc)$
Distributive law:	$a(b + c) = ab + ac$

These have been referred to during the discussion of the similarly named laws applicable to the union and intersection of sets. Most, but not all, of these laws also apply to matrix operations—the significant exception being the commutative law of multiplication.

Matrix Addition

Matrix addition is commutative as well as associative. This follows from the fact that matrix addition calls only for the addition of the corresponding elements of two matrices, and that the order in which each pair of corresponding elements is added is immaterial. In this context, incidentally, the subtraction operation $A - B$ can simply be regarded as the addition operation $A + (-B)$, and thus no separate discussion is necessary.

The commutative and associative laws can be stated as follows:

Commutative law $\quad A + B = B + A$

PROOF $\quad A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$

Example 1 Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$, we find that

$$A + B = B + A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

Associative law $\quad (A + B) + C = A + (B + C)$

PROOF
$$(A + B) + C = [a_{ij} + b_{ij}] + [c_{ij}] = [a_{ij} + b_{ij} + c_{ij}]$$
$$= [a_{ij}] + [b_{ij} + c_{ij}] = A + (B + C)$$

Example 2 Given $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we find that

$$(v_1 + v_2) - v_3 = \begin{bmatrix} 12 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

which is equal to

$$v_1 + (v_2 - v_3) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Applied to the linear combination of vectors $k_1 v_1 + \cdots + k_n v_n$, this law permits us to select any pair of terms for addition (or subtraction) first, instead of having to follow the sequence in which the n terms are listed.

Matrix Multiplication

Matrix multiplication is *not* commutative, that is,

$$AB \neq BA$$

As explained previously, even when AB is defined, BA may not be; but even if both products are defined, the general rule is still $AB \neq BA$.

Example 3 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$; then

$$AB = \begin{bmatrix} 1(0) + 2(6) & 1(-1) + 2(7) \\ 3(0) + 4(6) & 3(-1) + 4(7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 0(1) - 1(3) & 0(2) - 1(4) \\ 6(1) + 7(3) & 6(2) + 7(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Example 4 Let u' be 1×3 (a row vector); then the corresponding column vector u must be 3×1. The product $u'u$ will be 1×1, but the product uu' will be 3×3. Thus, obviously, $u'u \neq uu'$.

In view of the general rule $AB \neq BA$, the terms *premultiply* and *postmultiply* are often used to specify the order of multiplication. In the product AB, the matrix B is said to be *pre*multiplied by A, and A to be *post*multiplied by B.

There do exist interesting exceptions to the rule $AB \neq BA$, however. One such case is when A is a square matrix and B is an identity matrix. Another is when A is the inverse of B, that is, when $A = B^{-1}$. Both of these will be taken up again later. It should also be remarked here that the scalar multiplication of a matrix *does* obey the commutative law; thus

$$kA = Ak$$

if k is a scalar.

Although it is not in general commutative, matrix multiplication *is* associative.

Associative law $\quad (AB)C = A(BC) = ABC$

In forming the product ABC, the conformability condition must naturally be satisfied by each *adjacent* pair of matrices. If A is $m \times n$ and if C is $p \times q$, then conformability requires that B be $n \times p$:

$$\underset{(m\times n)}{A}\ \underset{(n\times p)}{B}\ \underset{(p\times q)}{C}$$

Note the dual appearance of n and p in the dimension indicators. If the conformability condition is met, the associative law states that any *adjacent* pair of matrices may be multiplied out first, provided that the product is duly inserted in the exact place of the original pair.

Example 5 If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, then

$$x'Ax = x'(Ax) = [x_1 \quad x_2]\begin{bmatrix} a_{11}x_1 \\ a_{22}x_2 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2$$

which is a "weighted" sum of squares, in contrast to the simple sum of squares given by $x'x$. Exactly the same result comes from

$$(x'A)x = [a_{11}x_1 \quad a_{22}x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2$$

Matrix multiplication is also distributive.

Distributive law $A(B + C) = AB + AC$ [premultiplication by A]
$(B + C)A = BA + CA$ [postmultiplication by A]

In each case, the conformability conditions for addition as well as for multiplication must, of course, be observed.

EXERCISE 4.4

1 Given $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix}$, verify that

(*a*) $(A + B) + C = A + (B + C)$

(*b*) $(A + B) - C = A + (B - C)$

2 The subtraction of a matrix B may be considered as the addition of the matrix $(-1)B$. Does the commutative law of addition permit us to state that $A - B = B - A$? If not, how would you correct the statement?

3 Test the associative law of multiplication with the following matrices:

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -8 & 0 & 7 \\ 1 & 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 7 & 1 \end{bmatrix}$$

4 Prove that for any two scalars g and k

(*a*) $k(A + B) = kA + kB$

(*b*) $(g + k)A = gA + kA$