4 Show that a *diagonal matrix*, i.e., a matrix of the form

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

can be idempotent only if each diagonal element is either 1 or 0. How many different numerical idempotent diagonal matrices of dimension $n \times n$ can be constructed altogether from the matrix above?

4.6 TRANSPOSES AND INVERSES

When the rows and columns of a matrix A are interchanged—so that its first row becomes the first column, and vice versa—we obtain the *transpose* of A, which is denoted by A' or A^T. The prime symbol is by no means new to us; it was used earlier to distinguish a row vector from a column vector. In the newly introduced terminology, a row vector x' constitutes the transpose of the column vector x. The superscript T in the alternative symbol is obviously shorthand for the word transpose.

Example 1 Given $\underset{(2\times3)}{A} = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $\underset{(2\times2)}{B} = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$, we can interchange the rows and columns and write

$$\underset{(3\times2)}{A'} = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix} \qquad \text{and} \qquad \underset{(2\times2)}{B'} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$$

By definition, if a matrix A is $m \times n$, then its transpose A' must be $n \times m$. An $n \times n$ square matrix, however, possesses a transpose with the same dimension.

Example 2 If $C = \begin{bmatrix} 9 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$, then

$$C' = \begin{bmatrix} 9 & 2 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$$

Here, the dimension of each transpose is identical with that of the original matrix.

In D', we also note the remarkable result that D' inherits not only the dimension of D but also the original array of elements! The fact that $D' = D$ is the result of the symmetry of the elements with reference to the principal diagonal. Considering the principal diagonal in D as a mirror, the elements located to its northeast are exact images of the elements to its southwest; hence the first row reads identically with the first column, and so forth. The matrix D exemplifies the special class of square matrices known as *symmetric matrices*. Another example of such a matrix is the identity matrix I, which, as a symmetric matrix, has the transpose $I' = I$.

Properties of Transposes

The following properties characterize transposes:

(4.9) $\quad (A')' = A$

(4.10) $\quad (A + B)' = A' + B'$

(4.11) $\quad (AB)' = B'A'$

The first says that the transpose of the transpose is the original matrix—a rather self-evident conclusion.

The second property may be verbally stated thus: the transpose of a sum is the sum of the transposes.

Example 3 If $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$, then

$$(A + B)' = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}' = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

and
$$A' + B' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

The third property is that the transpose of a product is the product of the transposes *in reverse order*. To appreciate the necessity for the reversed order, let us examine the dimension conformability of the two products on the two sides of (4.11). If we let A be $m \times n$ and B be $n \times p$, then AB will be $m \times p$, and $(AB)'$ will be $p \times m$. For equality to hold, it is necessary that the right-hand expression $B'A'$ be of the identical dimension. Since B' is $p \times n$ and A' is $n \times m$, the product $B'A'$ is indeed $p \times m$, as required. The dimension of $B'A'$ thus works out. Note that, on the other hand, the product $A'B'$ is not even defined unless $m = p$.

Example 4 Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$, we have

$$(AB)' = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}' = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

and
$$B'A' = \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

$A = m \times n$
$A' = n \times n$
$AB = m \times p$ $\quad B = n \times p.$
$(AB)' = p \times m$ $\quad B' = p \times n$
$B'A' = p \times m$

This verifies the property.

Inverses and Their Properties

For a given matrix A, the transpose A' is always derivable. On the other hand, its *inverse* matrix—another type of "derived" matrix—may or may not exist. The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition

$$AA^{-1} = A^{-1}A = I \tag{4.12}$$

That is, whether A is pre- or postmultiplied by A^{-1}, the product will be the same identity matrix. This is another exception to the rule that matrix multiplication is not commutative.

The following points are worth noting:

1. Not every square matrix has an inverse—squareness is a *necessary* condition, but *not* a *sufficient* condition, for the existence of an inverse. If a square matrix A has an inverse, A is said to be *nonsingular*; if A possesses no inverse, it is called a *singular* matrix.
2. If A^{-1} does exist, then the matrix A can be regarded as the inverse of A^{-1}, just as A^{-1} is the inverse of A. In short, A and A^{-1} are inverses of each other.
3. If A is $n \times n$, then A^{-1} must also be $n \times n$; otherwise it cannot be conformable for *both* pre- and postmultiplication. The identity matrix produced by the multiplication will also be $n \times n$.
4. If an inverse exists, then it is unique. To prove its uniqueness, let us suppose that B has been found to be an inverse for A, so that

   $$AB = BA = I$$

   Now assume that there is another matrix C such that $AC = CA = I$. By premultiplying both sides of $AB = I$ by C, we find that

   $$CAB = CI(= C) \qquad \text{[by (4.8)]}$$

   Since $CA = I$ by assumption, the preceding equation is reducible to

   $$IB = C \qquad \text{or} \qquad B = C$$

   That is, B and C must be one and the same inverse matrix. For this reason, we can speak of *the* (as against *an*) inverse of A.
5. The two parts of condition (4.12)—namely, $AA^{-1} = I$ and $A^{-1}A = I$—actually imply each other, so that satisfying either equation is sufficient to establish the inverse relationship between A and A^{-1}. To prove this, we should show that if $AA^{-1} = I$, and if there is a matrix B such that $BA = I$, then $B = A^{-1}$ (so that $BA = I$ must in effect be the equation $A^{-1}A = I$). Let us postmultiply both sides of the given equation $BA = I$ by A^{-1}; then

   $$(BA)A^{-1} = IA^{-1}$$

   $$B(AA^{-1}) = IA^{-1} \qquad \text{[associative law]}$$

   $$BI = IA^{-1} \qquad [AA^{-1} = I \text{ by assumption}]$$

Therefore, as required,

$$B = A^{-1} \qquad [\text{by } (4.8)]$$

Analogously, it can be demonstrated that, if $A^{-1}A = I$, then the only matrix C which yields $CA^{-1} = I$ is $C = A$.

Example 5 Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$; then, since the scalar multiplier ($\frac{1}{6}$) in B can be moved to the rear (commutative law), we can write

$$AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This establishes B as the inverse of A, and vice versa. The reverse multiplication, as expected, also yields the same identity matrix:

$$BA = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The following three properties of inverse matrices are of interest. If A and B are nonsingular matrices with dimension $n \times n$, then:

(4.13) $\quad (A^{-1})^{-1} = A$

(4.14) $\quad (AB)^{-1} = B^{-1}A^{-1}$

(4.15) $\quad (A')^{-1} = (A^{-1})'$

The first says that the inverse of an inverse is the original matrix. The second states that the inverse of a product is the product of the inverses *in reverse order*. And the last one means that the inverse of the transpose is the transpose of the inverse. Note that in these statements the existence of the inverses and the satisfaction of the conformability condition are presupposed.

The validity of (4.13) is fairly obvious, but let us prove (4.14) and (4.15). Given the product AB, let us find its inverse—call it C. From (4.12) we know that $CAB = I$; thus, postmultiplication of both sides by $B^{-1}A^{-1}$ will yield

(4.16) $\quad CABB^{-1}A^{-1} = IB^{-1}A^{-1} \; (= B^{-1}A^{-1})$ $\quad CAB B^{-1}A^{-1} = I B^{-1}A^{-1}$

But the left side is reducible to $\quad C = B^{-1}A^{-1}$

$$\begin{aligned} CA(BB^{-1})A^{-1} &= CAIA^{-1} && [\text{by } (4.12)] \\ &= CAA^{-1} = CI = C && [\text{by } (4.12) \text{ and } (4.8)] \end{aligned}$$

Substitution of this into (4.16) then tells us that $C = B^{-1}A^{-1}$ or, in other words, that the inverse of AB is equal to $B^{-1}A^{-1}$, as alleged. In this proof, the equation $AA^{-1} = A^{-1}A = I$ was utilized twice. Note that the application of this equation is permissible if and only if a matrix and its inverse are strictly adjacent to each other in a product. We may write $AA^{-1}B = IB = B$, but *never* $ABA^{-1} = B$.

The proof of (4.15) is as follows. Given A', let us find its inverse—call it D. By definition, we then have $DA' = I$. But we know that

$$(AA^{-1})' = I' = I$$

produces the same identity matrix. Thus we may write

$$\begin{aligned} DA' &= (AA^{-1})' \\ &= (A^{-1})'A' \qquad \text{[by (4.11)]} \end{aligned}$$

Postmultiplying both sides by $(A')^{-1}$, we obtain

$$DA'(A')^{-1} = (A^{-1})'A'(A')^{-1}$$

or

$$D = (A^{-1})' \qquad \text{[by (4.12)]}$$

Thus, the inverse of A' is equal to $(A^{-1})'$, as alleged.

In the proofs just presented, mathematical operations were performed on whole blocks of numbers. If those blocks of numbers had not been treated as mathematical entities (matrices), the same operations would have been much more lengthy and involved. The beauty of matrix algebra lies precisely in its simplification of such operations.

Inverse Matrix and Solution of Linear-Equation System

The application of the concept of inverse matrix to the solution of a simultaneous-equation system is immediate and direct. Referring to the equation system in (4.3), we pointed out earlier that it can be written in matrix notation as

$$(4.17) \qquad \underset{(3\times 3)}{A} \; \underset{(3\times 1)}{x} = \underset{(3\times 1)}{d}$$

where A, x, and d are as defined in (4.4). Now if the inverse matrix A^{-1} exists, the premultiplication of both sides of the equation (4.17) by A^{-1} will yield

$$A^{-1}Ax = A^{-1}d$$

or

$$(4.18) \qquad \underset{(3\times 1)}{x} = \underset{(3\times 3)}{A^{-1}} \; \underset{(3\times 1)}{d}$$

The left side of (4.18) is a column vector of variables, whereas the right-hand product is a column vector of certain known numbers. Thus, by definition of the equality of matrices or vectors, (4.18) shows the set of values of the variables that satisfy the equation system, i.e., the solution values. Furthermore, since A^{-1} is unique if it exists, $A^{-1}d$ must be a unique vector of solution values. We shall therefore write the x vector in (4.18) as $\bar{x}$, to indicate its status as a (unique) solution.

Methods of testing the existence of the inverse and of its calculation will be discussed in the next chapter. It may be stated here, however, that the inverse of the matrix A in (4.4) is

$$A^{-1} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

Thus (4.18) will turn out to be

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

which gives the solution: $\bar{x}_1 = 2$, $\bar{x}_2 = 3$, and $\bar{x}_3 = 1$.

The upshot is that, as one way of finding the solution of a linear-equation system $Ax = d$, where the coefficient matrix A is nonsingular, we can first find the inverse A^{-1}, and then postmultiply A^{-1} by the constant vector d. The product $A^{-1}d$ will then give the solution values of the variables.

EXERCISE 4.6

1 Given $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}$, find A', B', and C'. ⇒ Transpose $A' = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

2 Use the matrices given in the preceding problem to verify that

(a) $(A + B)' = A' + B'$ (b) $(AC)' = C'A'$

3 Generalize the result (4.11) to the case of a product of three matrices by proving that, for any conformable matrices A, B, and C, the equation $(ABC)' = C'B'A'$ holds.

4 Given the following four matrices, test whether any one of them is the inverse of another:

$$D = \begin{bmatrix} 1 & 12 \\ 0 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 6 & 8 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{3} \end{bmatrix} \quad G = \begin{bmatrix} 4 & -\frac{1}{2} \\ -3 & \frac{1}{2} \end{bmatrix}$$

5 Generalize the result (4.14) by proving that, for any conformable nonsingular matrices A, B, and C, the equation $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

6 Let $A = I - X(X'X)^{-1}X'$.

(a) Must A be square? Must $(X'X)$ be square? Must X be square?

(b) Show that matrix A is idempotent. [*Note:* If X' and X are not square, it is inappropriate to apply (4.14).]