CHAPTER

SIX

COMPARATIVE STATICS AND THE CONCEPT OF DERIVATIVE

The present and the two following chapters will be devoted to the methods of comparative-static analysis.

6.1 THE NATURE OF COMPARATIVE STATICS

Comparative statics, as the name suggests, is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. For purposes of such a comparison, we always start by assuming a given initial equilibrium state. In the isolated-market model, for example, such an initial equilibrium will be represented by a determinate price $\overline{P}$ and a corresponding quantity $\overline{Q}$. Similarly, in the simple national-income model of (3.23), the initial equilibrium will be specified by a determinate $\overline{Y}$ and a corresponding $\overline{C}$. Now if we let a disequilibrating change occur in the model—in the form of a variation in the value of some parameter or exogenous variable—the initial equilibrium will, of course, be upset. As a result, the various endogenous variables must undergo certain adjustments. If it is assumed that a new equilibrium state relevant to the new values of the data can be defined and attained, the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in comparative statics we again disregard the process of adjustment of the variables; we merely compare the initial (*pre*change) equilibrium state with the final (*post*change) equilibrium state. Also, we again preclude the possibility of instability of equilibrium, for we assume the new equilibrium to be attainable, just as we do for the old.

A comparative-static analysis can be either qualitative or quantitative in nature. If we are interested only in the question of, say, whether an increase in investment I_0 will increase or decrease the equilibrium income $\bar{Y}$, the analysis will be qualitative because the *direction* of change is the only matter considered. But if we are concerned with the *magnitude* of the change in $\bar{Y}$ resulting from a given change in I_0 (that is, the size of the investment multiplier), the analysis will obviously be quantitative. By obtaining a quantitative answer, however, we can automatically tell the direction of change from its algebraic sign. Hence the quantitative analysis always embraces the qualitative.

It should be clear that the problem under consideration is essentially one of finding a *rate of change*: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable. For this reason, the mathematical concept of *derivative* takes on preponderant significance in comparative statics, because that concept—the most fundamental one in the branch of mathematics known as *differential calculus* — is directly concerned with the notion of rate of change! Later on, moreover, we shall find the concept of derivative to be of extreme importance for optimization problems as well.

6.2 RATE OF CHANGE AND THE DERIVATIVE

Even though our present context is concerned only with the rates of change of the equilibrium values of the variables in a model, we may carry on the discussion in a more general manner by considering the rate of change of any variable y in response to a change in another variable x, where the two variables are related to each other by the function

$$y = f(x)$$

Applied in the comparative-static context, the variable y will represent the equilibrium value of an endogenous variable, and x will be some parameter. Note that, for a start, we are restricting ourselves to the simple case where there is only a single parameter or exogenous variable in the model. Once we have mastered this simplified case, however, the extension to the case of more parameters will prove relatively easy.

The Difference Quotient

Since the notion of "change" figures prominently in the present context, a special symbol is needed to represent it. When the variable x changes from the value x_0 to a new value x_1, the change is measured by the difference $x_1 - x_0$. Hence, using the symbol Δ (the Greek capital delta, for "difference") to denote the change, we write $\Delta x = x_1 - x_0$. Also needed is a way of denoting the value of the function $f(x)$ at various values of x. The standard practice is to use the notation $f(x_i)$ to represent the value of $f(x)$ when $x = x_i$. Thus, for the function $f(x) = 5 + x^2$, we have $f(0) = 5 + 0^2 = 5$; and similarly, $f(2) = 5 + 2^2 = 9$, etc.

When x changes from an initial value x_0 to a new value $(x_0 + \Delta x)$, the value of the function $y = f(x)$ changes from $f(x_0)$ to $f(x_0 + \Delta x)$. The change in y per unit of change in x can be represented by the *difference quotient*

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{6.1}$$

This quotient, which measures the average rate of change of y, can be calculated if we know the initial value of x, or x_0, and the magnitude of change in x, or Δx. That is, $\Delta y/\Delta x$ is a function of x_0 and Δx.

Example 1 Given $y = f(x) = 3x^2 - 4$, we can write:

$$f(x_0) = 3(x_0)^2 - 4 \qquad f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$$

Therefore, the difference quotient is

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x} = \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} \\ &= 6x_0 + 3\Delta x \end{aligned} \tag{6.2}$$

which can be evaluated if we are given x_0 and Δx. Let $x_0 = 3$ and $\Delta x = 4$; then the average rate of change of y will be $6(3) + 3(4) = 30$. This means that, on the average, as x changes from 3 to 7, the change in y is 30 units per unit change in x.

The Derivative

Frequently, we are interested in the rate of change of y when Δx is very small. In such a case, it is possible to obtain an approximation of $\Delta y/\Delta x$ by dropping all the terms in the difference quotient involving the expression Δx. In (6.2), for instance, if Δx is very small, we may simply take the term $6x_0$ on the right as an approximation of $\Delta y/\Delta x$. The smaller the value of Δx, of course, the closer is the approximation to the true value of $\Delta y/\Delta x$.

As Δx approaches zero (meaning that it gets closer and closer to, but never actually reaches, zero), $(6x_0 + 3\Delta x)$ will approach the value $6x_0$, and by the same token, $\Delta y/\Delta x$ will approach $6x_0$ also. Symbolically, this fact is expressed either by the statement $\Delta y/\Delta x \to 6x_0$ as $\Delta x \to 0$, or by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x_0 + 3\Delta x) = 6x_0 \tag{6.3}$$

where the symbol $\lim_{\Delta x \to 0}$ is read: "The limit of... as Δx approaches 0." If, as $\Delta x \to 0$, the limit of the difference quotient $\Delta y/\Delta x$ exists, that limit is identified as the derivative of the function $y = f(x)$.

Several points should be noted about the derivative. First, a derivative is a *function*; in fact, in this usage the word *derivative* really means a derived function. The original function $y = f(x)$ is a *primitive function*, and the derivative is another function derived from it. Whereas the difference quotient is a function of x_0 and Δx, you should observe—from (6.3), for instance—that the derivative is a function of x_0 only. This is because Δx is already compelled to approach zero, and therefore it should not be regarded as another variable in the function. Let us also add that so far we have used the subscripted symbol x_0 only in order to stress the fact that a change in x must start from some specific value of x. Now that this is understood, we may delete the subscript and simply state that the derivative, like the primitive function, is itself a function of the independent variable x. That is, for each value of x, there is a unique corresponding value for the derivative function.

Second, since the derivative is merely a limit of the difference quotient, which measures a rate of change of y, the derivative must of necessity also be a measure of some rate of change. In view of the fact that the change in x envisaged in the derivative concept is infinitesimal (that is, $\Delta x \to 0$), however, the rate measured by the derivative is in the nature of an *instantaneous* rate of change.

Third, there is the matter of notation. Derivative functions are commonly denoted in two ways. Given a primitive function $y = f(x)$, one way of denoting its derivative (if it exists) is to use the symbol $f'(x)$, or simply f'; this notation is attributed to the mathematician Lagrange. The other common notation is dy/dx, devised by the mathematician Leibniz. [Actually there is a third notation, Dy, or $Df(x)$, but we shall not use it in the following discussion.] The notation $f'(x)$, which resembles the notation for the primitive function $f(x)$, has the advantage of conveying the idea that the derivative is itself a function of x. The reason for expressing it as $f'(x)$—rather than, say, $\phi(x)$—is to emphasize that the function f' is derived from the primitive function f. The alternative notation, dy/dx, serves instead to emphasize that the value of a derivative measures a rate of change. The letter d is the counterpart of the Greek Δ, and dy/dx differs from $\Delta y/\Delta x$ chiefly in that the former is the limit of the latter as Δx approaches zero. In the subsequent discussion, we shall use both of these notations, depending on which seems the more convenient in a particular context.

Using these two notations, we may define the derivative of a given function $y = f(x)$ as follows:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 2 Referring to the function $y = 3x^2 - 4$ again, we have shown its difference quotient to be (6.2), and the limit of that quotient to be (6.3). On the basis of the latter, we may now write (replacing x_0 with x):

$$\frac{dy}{dx} = 6x \qquad \text{or} \qquad f'(x) = 6x$$

Note that different values of x will give the derivative correspondingly different values. For instance, when $x = 3$, we have $f'(x) = 6(3) = 18$; but when $x = 4$, we find that $f'(4) = 6(4) = 24$.

EXERCISE 6.2

1 Given the function $y = 4x^2 + 9$:
(*a*) Find the difference quotient as a function of x and Δx. (Use x in lieu of x_0).
(*b*) Find the derivative dy/dx.
(*c*) Find $f'(3)$ and $f'(4)$.

2 Given the function $y = 5x^2 - 4x$:
(*a*) Find the difference quotient as a function of x and Δx.
(*b*) Find the derivative dy/dx.
(*c*) Find $f'(2)$ and $f'(3)$.

3 Given the function $y = 5x - 2$:
(*a*) Find the difference quotient $\Delta y/\Delta x$. What type of function is it?
(*b*) Since the expression Δx does not appear in the function $\Delta y/\Delta x$ above, does it make any difference to the value of $\Delta y/\Delta x$ whether Δx is large or small? Consequently, what is the limit of the difference quotient as Δx approaches zero?

6.3 THE DERIVATIVE AND THE SLOPE OF A CURVE

Elementary economics tells us that, given a total-cost function $C = f(Q)$, where C denotes total cost and Q the output, the marginal cost (MC) is defined as the change in total cost resulting from a unit increase in output; that is, $\text{MC} = \Delta C/\Delta Q$. It is understood that ΔQ is an extremely small change. For the case of a product that has discrete units (integers only), a change of one unit is the smallest change possible; but for the case of a product whose quantity is a continuous variable, ΔQ will refer to an infinitesimal change. In this latter case, it is well known that the marginal cost can be measured by the slope of the total-cost curve. But the slope of the total-cost curve is nothing but the limit of the ratio $\Delta C/\Delta Q$, when ΔQ approaches zero. Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative. Both have to do with the "marginal" notion so extensively used in economics.

In Fig. 6.1, we have drawn a total-cost curve C, which is the graph of the (primitive) function $C = f(Q)$. Suppose that we consider Q_0 as the initial output level from which an increase in output is measured, then the relevant point on the cost curve will be A. If output is to be raised to $Q_0 + \Delta Q = Q_2$, the total cost will be increased from C_0 to $C_0 + \Delta C = C_2$; thus $\Delta C/\Delta Q = (C_2 - C_0)/(Q_2 - Q_0)$. Geometrically, this is the ratio of two line segments, EB/AE, or the *slope* of the line AB. This particular ratio measures an average rate of change—the *average*

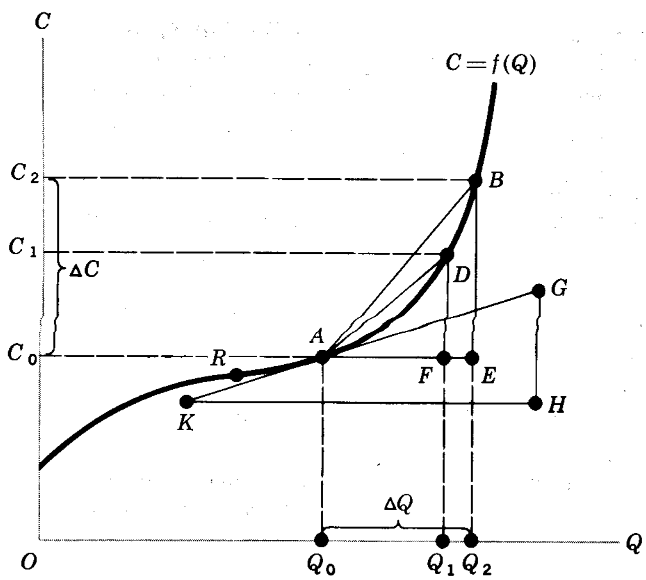

Figure 6.1

marginal cost for the particular ΔQ pictured—and represents a difference quotient. As such, it is a function of the initial value Q_0 and the amount of change ΔQ.

What happens when we vary the magnitude of ΔQ? If a smaller output increment is contemplated (say, from Q_0 to Q_1 only), then the average marginal cost will be measured by the slope of the line AD instead. Moreover, as we reduce the output increment further and further, flatter and flatter lines will result until, in the limit (as $\Delta Q \to 0$), we obtain the line KG (which is the *tangent line* to the cost curve at point A) as the relevant line. The slope of $KG(= HG/KH)$ measures the slope of the total-cost curve at point A and represents the limit of $\Delta C/\Delta Q$, as $\Delta Q \to 0$, when initial output is at $Q = Q_0$. Therefore, in terms of the derivative, the slope of the $C = f(Q)$ curve at point A corresponds to the particular derivative value $f'(Q_0)$.

What if the initial output level is changed from Q_0 to, say, Q_2? In that case, point B on the curve will replace point A as the relevant point, and the slope of the curve at the new point B will give us the derivative value $f'(Q_2)$. Analogous results are obtainable for alternative initial output levels. In general, the derivative $f'(Q)$—a function of Q—will vary as Q changes.

6.4 THE CONCEPT OF LIMIT

The derivative dy/dx has been defined as the limit of the difference quotient $\Delta y/\Delta x$ as $\Delta x \to 0$. If we adopt the shorthand symbols $q \equiv \Delta y/\Delta x$ (q for quotient) and $v \equiv \Delta x$ (v for variation), we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

In view of the fact that the derivative concept relies heavily on the notion of limit, it is imperative that we get a clear idea about that notion.

Left-Side Limit and Right-Side Limit

The concept of limit is concerned with the question: "What value does one variable (say, q) approach as another variable (say, v) approaches a specific value (say, zero)?" In order for this question to make sense, q must, of course, be a function of v; say, $q = g(v)$. Our immediate interest is in finding the limit of q as $v \to 0$, but we may just as easily explore the more general case of $v \to N$, where N is any finite real number. Then, $\lim_{v \to 0} q$ will be merely a special case of $\lim_{v \to N} q$ where $N = 0$. In the course of the discussion, we shall actually also consider the limit of q as $v \to +\infty$ (plus infinity) or as $v \to -\infty$ (minus infinity).

When we say $v \to N$, the variable v can approach the number N either from values greater than N, or from values less than N. If, as $v \to N$ from the left side (from values less than N), q approaches a finite number L, we call L the *left-side limit* of q. On the other hand, if L is the number that q tends to as $v \to N$ from the right side (from values greater than N), we call L the *right-side limit* of q. The left- and right-side limits may or may not be equal.

The left-side limit of q is symbolized by $\lim_{v \to N^-} q$ (the minus sign signifies from values less than N), and the right-side limit is written as $\lim_{v \to N^+} q$. When—and only when—the two limits have a common finite value (say, L), we consider the limit of q to exist and write it as $\lim_{v \to N} q = L$. Note that L must be a *finite* number. If we have the situation of $\lim_{v \to N} q = \infty$ (or $-\infty$), we shall consider q to possess *no* limit, because $\lim_{v \to N} q = \infty$ means that $q \to \infty$ as $v \to N$, and if q will assume *ever-increasing* values as v tends to N, it would be contradictory to say that q has a limit. As a convenient way of expressing the fact that $q \to \infty$ as $v \to N$, however, people do indeed write $\lim_{v \to N} q = \infty$ and speak of q as having an "infinite limit."

In certain cases, only the limit of one side needs to be considered. In taking the limit of q as $v \to +\infty$, for instance, only the left-side limit of q is relevant, because v can approach $+\infty$ only from the left. Similarly, for the case of $v \to -\infty$, only the right-side limit is relevant. Whether the limit of q exists in these cases will depend only on whether q approaches a finite value as $v \to +\infty$, or as $v \to -\infty$.

It is important to realize that the symbol ∞ (infinity) is not a number, and therefore it cannot be subjected to the usual algebraic operations. We cannot have

$3 + \infty$ or $1/\infty$; nor can we write $q = \infty$, which is not the same as $q \to \infty$. However, it is acceptable to express the *limit* of q as " = " (as against $\to$) ∞, for this merely indicates that $q \to \infty$.

Graphical Illustrations

Let us illustrate, in Fig. 6.2, several possible situations regarding the limit of a function $q = g(v)$.

Figure 6.2*a* shows a smooth curve. As the variable v tends to the value N from *either* side on the horizontal axis, the variable q tends to the value L. In this case, the left-side limit is identical with the right-side limit; therefore we can write $\lim_{v \to N} q = L$.

The curve drawn in Fig. 6.2*b* is not smooth; it has a sharp turning point directly above the point N. Nevertheless, as v tends to N from either side, q again tends to an identical value L. The limit of q again exists and is equal to L.

Figure 6.2*c* shows what is known as a *step function*.* In this case, as v tends to N, the left-side limit of q is L_1, but the right-side limit is L_2, a different number. Hence, q does not have a limit as $v \to N$.

Lastly, in Fig. 6.2*d*, as v tends to N, the left-side limit of q is $-\infty$, whereas the right-side limit is $+\infty$, because the two parts of the (hyperbolic) curve will fall and rise indefinitely while approaching the broken vertical line as an asymptote. Again, $\lim_{v \to N} q$ does not exist. On the other hand, if we are considering a different sort of limit in diagram *d*, namely, $\lim_{v \to +\infty} q$, then only the left-side limit has relevance, and we do find that limit to exist: $\lim_{v \to +\infty} q = M$. Analogously, you can verify that $\lim_{v \to -\infty} q = M$ as well.

It is also possible to apply the concepts of left-side and right-side limits to the discussion of the marginal cost in Fig. 6.1. In that context, the variables q and v will refer, respectively, to the quotient $\Delta C/\Delta Q$ and to the magnitude of ΔQ, with all changes being measured from point A on the curve. In other words, q will refer to the slope of such lines as AB, AD, and KG, whereas v will refer to the length of such lines as Q_0Q_2 (= line AE) and Q_0Q_1 (= line AF). We have already seen that, as v approaches zero from a positive value, q will approach a value equal to the slope of line KG. Similarly, we can establish that, if ΔQ approaches zero from

* This name is easily explained by the shape of the curve. But step functions can be expressed algebraically, too. The one illustrated in Fig. 6.2*c* can be expressed by the equation

$$q = \begin{cases} L_1 & (\text{for } 0 \le v < N) \\ L_2 & (\text{for } N \le v) \end{cases}$$

Note that, in each subset of its domain described above, the function appears as a distinct constant function, which constitutes a "step" in the graph.

In economics, step functions can be used, for instance, to show the various prices charged for different quantities purchased (the curve shown in Fig. 6.2*c* pictures *quantity discount*) or the various tax rates applicable to different income brackets.

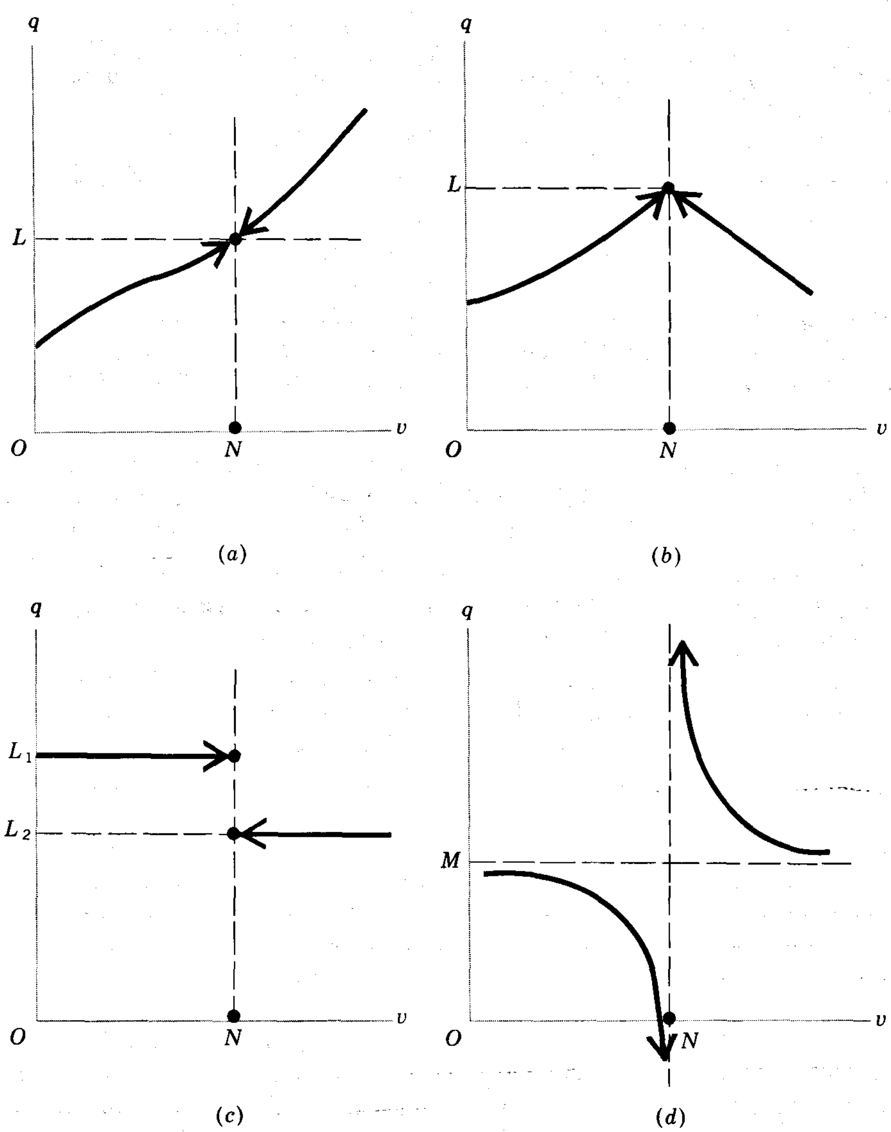

Figure 6.2

a negative value (i.e., as the *decrease* in output becomes less and less), the quotient $\Delta C/\Delta Q$, as measured by the slope of such lines as *RA* (not drawn), will also approach a value equal to the slope of line *KG*. Indeed, the situation here is very much akin to that illustrated in Fig. 6.2*a*. Thus the slope of *KG* in Fig. 6.1 (the counterpart of *L* in Fig. 6.2) is indeed the limit of the quotient q as v tends to zero, and as such it gives us the marginal cost at the output level $Q = Q_0$.

Evaluation of a Limit

Let us now illustrate the algebraic evaluation of a limit of a given function $q = g(v)$.

Example 1 Given $q = 2 + v^2$, find $\lim_{v \to 0} q$. To take the left-side limit, we substitute the series of negative values $-1, -\frac{1}{10}, -\frac{1}{100}, \ldots$ (in that order) for v and find that $(2 + v^2)$ will decrease steadily and approach 2 (because v^2 will gradually approach 0). Next, for the right-side limit, we substitute the series of positive values $1, \frac{1}{10}, \frac{1}{100}, \ldots$ (in that order) for v and find the same limit as before. Inasmuch as the two limits are identical, we consider the limit of q to exist and write $\lim_{v \to 0} q = 2$.

It is tempting to regard the answer just obtained as the outcome of setting $v = 0$ in the equation $q = 2 + v^2$, but this temptation should in general be resisted. In evaluating $\lim_{v \to N} q$, we only let v *tend to* N but, as a rule, do not let $v = N$. Indeed, we can quite legitimately speak of the limit of q as $v \to N$, even if N is *not* in the domain of the function $q = g(v)$. In this latter case, if we try to set $v = N$, q will clearly be undefined.

Example 2 Given $q = (1 - v^2)/(1 - v)$, find $\lim_{v \to 1} q$. Here, $N = 1$ is not in the domain of the function, and we cannot set $v = 1$ because that would involve division by zero. Moreover, even the limit-evaluation procedure of letting $v \to 1$, as used in Example 1, will cause difficulty, for the denominator $(1 - v)$ will approach zero when $v \to 1$, and we will still have no way of performing the division in the limit.

One way out of this difficulty is to try to transform the given ratio to a form in which v will not appear in the denominator. Since $v \to 1$ implies that $v \neq 1$, so that $(1 - v)$ is nonzero, it is legitimate to divide the expression $(1 - v^2)$ by $(1 - v)$, and write*

$$q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1)$$

* The division can be performed, as in the case of numbers, in the following manner:

$$\begin{array}{r|l} & 1 + v \\ \hline 1 - v & 1 \quad\; - v^2 \\ & 1 - v \\ \hline & \quad v - v^2 \\ & \quad \underline{\underline{v - v^2}} \end{array}$$

Alternatively, we may resort to factoring as follows:

$$\frac{1 - v^2}{1 - v} = \frac{(1 + v)(1 - v)}{1 - v} = 1 + v \qquad (v \neq 1)$$

In this new expression for q, there is no longer a denominator with v in it. Since $(1 + v) \to 2$ as $v \to 1$ from *either* side, we may then conclude that $\lim_{v \to 1} q = 2$.

Example 3 Given $q = (2v + 5)/(v + 1)$, find $\lim_{v \to +\infty} q$. The variable v again appears in *both* the numerator and the denominator. If we let $v \to +\infty$ in both, the result will be a ratio between two infinitely large numbers, which does not have a clear meaning. To get out of the difficulty, we try this time to transform the given ratio to a form in which the variable v will not appear in the numerator.* This, again, can be accomplished by dividing out the given ratio. Since $(2v + 5)$ is not evenly divisible by $(v + 1)$, however, the result will contain a remainder term as follows:

$$q = \frac{2v + 5}{v + 1} = 2 + \frac{3}{v + 1}$$

But, at any rate, this new expression for q no longer has a numerator with v in it. Noting that the remainder $3/(v + 1) \to 0$ as $v \to +\infty$, we can then conclude that $\lim_{v \to +\infty} q = 2$.

There also exist several useful theorems on the evaluation of limits. These will be discussed in Sec. 6.6.

Formal View of the Limit Concept

The above discussion should have conveyed some general ideas about the concept of limit. Let us now give it a more precise definition. Since such a definition will make use of the concept of *neighborhood* of a point on a line (in particular, a specific number as a point on the line of real numbers), we shall first explain the latter term.

For a given number L, there can always be found a number $(L - a_1) < L$ and another number $(L + a_2) > L$, where a_1 and a_2 are some arbitrary positive numbers. The set of all numbers falling between $(L - a_1)$ and $(L + a_2)$ is called the *interval* between those two numbers. If the numbers $(L - a_1)$ and $(L + a_2)$ are included in the set, the set is a *closed interval*; if they are excluded, the set is an *open interval*. A closed interval between $(L - a_1)$ and $(L + a_2)$ is denoted by the bracketed expression

$$[L - a_1, L + a_2] \equiv \{q \mid L - a_1 \le q \le L + a_2\}$$

and the corresponding *open* interval is denoted with parentheses:

$$(L - a_1, L + a_2) \equiv \{q \mid L - a_1 < q < L + a_2\} \tag{6.4}$$

* Note that, unlike the $v \to 0$ case, where we want to take v out of the *denominator* in order to avoid division by zero, the $v \to \infty$ case is better served by taking v out of the *numerator*. As $v \to \infty$, an expression containing v in the numerator will become infinite but an expression with v in the denominator will, more conveniently for us, approach zero and quietly vanish from the scene.

Thus, [ ] relate to the weak inequality sign $\leq$, whereas ( ) relate to the strict inequality sign $<$. But in both types of intervals, the smaller number $(L - a_1)$ is always listed first. Later on, we shall also have occasion to refer to *half-open and half-closed* intervals such as (3, 5] and [6, ∞), which have the following meanings:

$$(3,5] \equiv \{x \mid 3 < x \leq 5\} \qquad [6,\infty) \equiv \{x \mid 6 \leq x < \infty\}$$

Now we may define a *neighborhood* of L to be an open interval as defined in (6.4), which is an interval "covering" the number L.* Depending on the magnitudes of the arbitrary numbers a_1 and a_2, it is possible to construct various neighborhoods for the given number L. Using the concept of neighborhood, the limit of a function may then be defined as follows:

> As v approaches a number N, the limit of $q = g(v)$ is the number L, if, for every neighborhood of L that can be chosen, *however small*, there can be found a corresponding neighborhood of N (excluding the point $v = N$) in the domain of the function such that, for every value of v in that N-neighborhood, its image lies in the chosen L-neighborhood.

This statement can be clarified with the help of Fig. 6.3, which resembles Fig. 6.2a. From what was learned about the latter figure, we know that $\lim_{v \to N} q = L$ in Fig. 6.3. Let us show that L does indeed fulfill the new definition of a limit. As the first step, select an arbitrary small neighborhood of L, say, $(L - a_1,\ L + a_2)$. (This should have been made even smaller, but we are keeping it relatively large to facilitate exposition.) Now construct a neighborhood of N, say, $(N - b_1,\ N + b_2)$, such that the two neighborhoods (when extended into quadrant I) will together define a rectangle (shaded in diagram) with two of its corners lying on the given curve. It can then be verified that, for every value of v in this neighborhood of N (not counting $v = N$), the corresponding value of $q = g(v)$ lies in the chosen neighborhood of L. In fact, no matter how *small* an L-neighborhood we choose, a (correspondingly small) N-neighborhood can be found with the property just cited. Thus L fulfills the definition of a limit, as was to be demonstrated.

We can also apply the above definition to the step function of Fig. 6.2c in order to show that neither L_1 nor L_2 qualifies as $\lim_{v \to N} q$. If we choose a very small neighborhood of L_1—say, just a hair's width on each side of L_1—then, no matter what neighborhood we pick for N, the rectangle associated with the two neighborhoods cannot possibly enclose the lower step of the function. Consequently, for any value of $v > N$, the corresponding value of q (located on the lower step) will not be in the neighborhood of L_1, and thus L_1 fails the test for a limit. By similar reasoning, L_2 must also be dismissed as a candidate for $\lim_{v \to N} q$. In fact, in this case no limit exists for q as $v \to N$.

* The identification of an open interval as the neighborhood of a point is valid only when we are considering a point on a line (one-dimensional space). In the case of a point in a plane (two-dimensional space), its neighborhood must be thought of as an area, say, a circular area around the point.

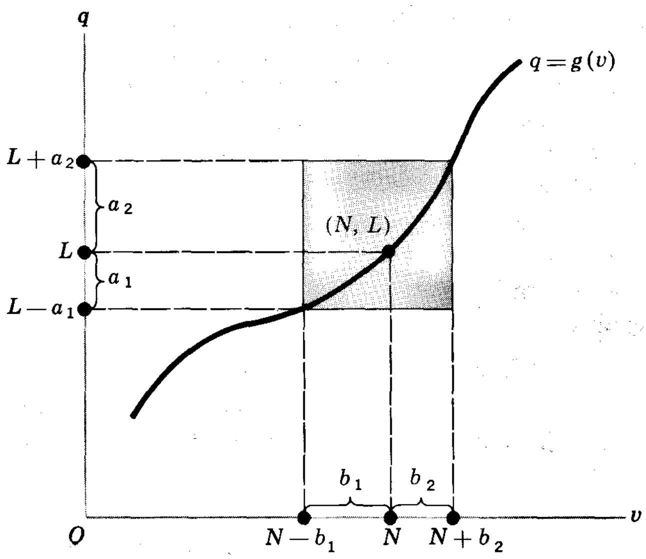

Figure 6.3

The fulfillment of the definition can also be checked algebraically rather than by graph. For instance, consider again the function

$$(6.5) \qquad q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1)$$

It has been found in Example 2 that $\lim_{v \to 1} q = 2$; thus, here we have $N = 1$ and $L = 2$. To verify that $L = 2$ is indeed the limit of q, we must demonstrate that, for every chosen neighborhood of L, $(2 - a_1, 2 + a_2)$, there exists a neighborhood of N, $(1 - b_1, 1 + b_2)$, such that, whenever v is in this neighborhood of N, q must be in the chosen neighborhood of L. This means essentially that, for given values of a_1 and a_2, however small, two numbers b_1 and b_2 must be found such that, whenever the inequality

$$(6.6) \qquad 1 - b_1 < v < 1 + b_2 \qquad (v \neq 1)$$

is satisfied, another inequality of the form

$$(6.7) \qquad 2 - a_1 < q < 2 + a_2$$

must also be satisfied. To find such a pair of numbers b_1 and b_2, let us first rewrite (6.7) by substituting (6.5):

$$(6.7') \qquad 2 - a_1 < 1 + v < 2 + a_2$$

This, in turn, can be transformed into the inequality

$$(6.7'') \qquad 1 - a_1 < v < 1 + a_2$$

A comparison of (6.7″)—a variant of (6.7)—with (6.6) suggests that if we choose the two numbers b_1 and b_2 to be $b_1 = a_1$ and $b_2 = a_2$, the two inequalities (6.6) and (6.7) will always be satisfied simultaneously. Thus the neighborhood of N, $(1 - b_1, 1 + b_2)$, as required in the definition of a limit, can indeed be found for the case of $L = 2$, and this establishes $L = 2$ as the limit.

Let us now utilize the definition of a limit in the opposite way, to show that another value (say, 3) cannot qualify as $\lim_{v \to 1} q$ for the function in (6.5). If 3 were that limit, it would have to be true that, for every chosen neighborhood of 3, $(3 - a_1, 3 + a_2)$, there exists a neighborhood of 1, $(1 - b_1, 1 + b_2)$, such that, whenever v is in the latter neighborhood, q must be in the former neighborhood. That is, whenever the inequality

$$1 - b_1 < v < 1 + b_2$$

is satisfied, another inequality of the form

$$3 - a_1 < 1 + v < 3 + a_2$$

$$\text{or} \qquad 2 - a_1 < v < 2 + a_2$$

must also be satisfied. The *only* way to achieve this result is to choose $b_1 = a_1 - 1$ and $b_2 = a_2 + 1$. This would imply that the neighborhood of 1 is to be the open interval $(2 - a_1, 2 + a_2)$. According to the definition of a limit, however, a_1 and a_2 can be made arbitrarily small, say, $a_1 = a_2 = 0.1$. In that case, the last-mentioned interval will turn out to be (1.9, 2.1) which lies entirely to the right of the point $v = 1$ on the horizontal axis and, hence, does not even qualify as a neighborhood of 1. Thus the definition of a limit cannot be satisfied by the number 3. A similar procedure can be employed to show that *any* number other than 2 will contradict the definition of a limit in the present case.

In general, if one number satisfies the definition of a limit of q as $v \to N$, then no other number can. If a limit exists, it will be unique.

EXERCISE 6.4

1 Given the function $q = (v^2 + v - 56)/(v - 7)$, $(v \neq 7)$, find the left-side limit and the right-side limit of q as v approaches 7. Can we conclude from these answers that q has a limit as v approaches 7?

2 Given $q = [(v + 2)^3 - 8]/v$, $(v \neq 0)$, find:
(*a*) $\lim_{v \to 0} q$ (*b*) $\lim_{v \to 2} q$ (*c*) $\lim_{v \to a} q$

3 Given $q = 5 - 1/v$, $(v \neq 0)$, find:
(*a*) $\lim_{v \to +\infty} q$ (*b*) $\lim_{v \to -\infty} q$

4 Use Fig. 6.3 to show that we *cannot* consider the number $(L + a_2)$ as the limit of q as v tends to N.