6.6 LIMIT THEOREMS

Our interest in rates of change led us to the consideration of the concept of derivative, which, being in the nature of the limit of a difference quotient, in turn prompted us to study questions of the existence and evaluation of a limit. The basic process of limit evaluation, as illustrated in Sec. 6.4, involves letting the variable v approach a particular number (say, N) and observing the value which q approaches. When actually evaluating the limit of a function, however, we may draw upon certain established limit theorems, which can materially simplify the task, especially for complicated functions.

Theorems Involving a Single Function

When a single function $q = g(v)$ is involved, the following theorems are applicable.

Theorem I If $q = av + b$, then $\lim_{v \to N} q = aN + b$ (a and b are constants).

Example 1 Given $q = 5v + 7$, we have $\lim_{v \to 2} q = 5(2) + 7 = 17$. Similarly, $\lim_{v \to 0} q = 5(0) + 7 = 7$.

Theorem II If $q = g(v) = b$, then $\lim_{v \to N} q = b$.

This theorem, which says that the limit of a constant function is the constant in that function, is merely a special case of Theorem I, with $a = 0$. (You have already encountered an example of this case in Exercise 6.2-3.)

Theorem III If $q = v$, then $\lim_{v \to N} q = N$.

If $q = v^k$, then $\lim_{v \to N} q = N^k$.

Example 2 Given $q = v^3$, we have $\lim_{v \to 2} q = (2)^3 = 8$.

You may have noted that, in the above three theorems, what is done to find the limit of q as $v \to N$ is indeed to let $v = N$. But these are special cases, and they do not vitiate the general rule that "$v \to N$" does not mean "$v = N$."

Theorems Involving Two Functions

If we have two functions of the same independent variable v, $q_1 = g(v)$ and $q_2 = h(v)$, and if *both* functions possess limits as follows:

$$\lim_{v \to N} q_1 = L_1 \qquad \lim_{v \to N} q_2 = L_2$$

where L_1 and L_2 are two *finite* numbers, the following theorems are applicable.

Theorem IV (sum-difference limit theorem)

$$\lim_{v \to N} (q_1 \pm q_2) = L_1 \pm L_2$$

The limit of a sum (difference) of two functions is the sum (difference) of their respective limits.

In particular, we note that

$$\lim_{v \to N} 2q_1 = \lim_{v \to N} (q_1 + q_1) = L_1 + L_1 = 2L_1$$

which is in line with Theorem I.

Theorem V (product limit theorem)

$$\lim_{v \to N} (q_1 q_2) = L_1 L_2$$

The limit of a product of two functions is the product of their limits.

Applied to the square of a function, this gives

$$\lim_{v \to N} (q_1 q_1) = L_1 L_1 = L_1^2$$

which is in line with Theorem III.

Theorem VI (quotient limit theorem)

$$\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2} \qquad (L_2 \neq 0)$$

The limit of a quotient of two functions is the quotient of their limits. Naturally, the limit L_2 is restricted to be nonzero; otherwise the quotient is undefined.

Example 3 Find $\lim_{v \to 0} (1 + v)/(2 + v)$. Since we have here $\lim_{v \to 0} (1 + v) = 1$ and $\lim_{v \to 0} (2 + v) = 2$, the desired limit is $\frac{1}{2}$.

Remember that L_1 and L_2 represent finite numbers; otherwise these theorems do not apply. In the case of Theorem VI, furthermore, L_2 must be nonzero as well. If these restrictions are not satisfied, we must fall back on the method of limit evaluation illustrated in Examples 2 and 3 in Sec. 6.4, which relate to the cases, respectively, of L_2 being zero and of L_2 being infinite.

Limit of a Polynomial Function

With the above limit theorems at our disposal, we can easily evaluate the limit of any polynomial function

$$(6.11) \qquad q = g(v) = a_0 + a_1 v + a_2 v^2 + \cdots + a_n v^n$$

as v tends to the number N. Since the limits of the separate terms are, respectively,

$$\lim_{v \to N} a_0 = a_0 \qquad \lim_{v \to N} a_1 v = a_1 N \qquad \lim_{v \to N} a_2 v^2 = a_2 N^2 \qquad \text{(etc.)}$$

the limit of the polynomial function is (by the sum limit theorem)

$$\lim_{v \to N} q = a_0 + a_1 N + a_2 N^2 + \cdots + a_n N^n \tag{6.12}$$

This limit is also, we note, actually equal to $g(N)$, that is, equal to the value of the function in (6.11) when $v = N$. This particular result will prove important in discussing the concept of *continuity* of the polynomial function.

EXERCISE 6.6

1 Find the limits of the function $q = 8 - 9v + v^2$:
(*a*) As $v \to 0$ (*b*) As $v \to 3$ (*c*) As $v \to -1$

2 Find the limits of $q = (v + 2)(v - 3)$:
(*a*) As $v \to -1$ (*b*) As $v \to 0$ (*c*) As $v \to 4$

3 Find the limits of $q = (3v + 5)/(v + 2)$:
(*a*) As $v \to 0$ (*b*) As $v \to 5$ (*c*) As $v \to -1$

6.7 CONTINUITY AND DIFFERENTIABILITY OF A FUNCTION

The preceding discussion of the concept of limit and its evaluation can now be used to define the continuity and differentiability of a function. These notions bear directly on the derivative of the function, which is what interests us.

Continuity of a Function

When a function $q = g(v)$ possesses a limit as v tends to the point N in the domain, and when this limit is also equal to $g(N)$—that is, equal to the value of the function at $v = N$—the function is said to be *continuous* at N. As stated above, the term *continuity* involves no less than three requirements: (1) the point N must be in the domain of the function; i.e., $g(N)$ is defined; (2) the function must have a limit as $v \to N$; i.e., $\lim_{v \to N} g(v)$ exists; and (3) that limit must be equal in value to $g(N)$; i.e., $\lim_{v \to N} g(v) = g(N)$.

It is important to note that while—in discussing the limit of the curve in Fig. 6.3—the point (N, L) was excluded from consideration, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point (N, L) must be on the graph of the function before the function can be considered as continuous at point N.

Let us check whether the functions shown in Fig. 6.2 are continuous. In diagram *a*, all three requirements are met at point N. Point N is in the domain; q

has the limit L as $v \to N$; and the limit L happens also to be the value of the function at N. Thus, the function represented by that curve is continuous at N. The same is true of the function depicted in Fig. 6.2*b*, since L is the limit of the function as v approaches the value N in the domain, and since L is also the value of the function at N. This last graphic example should suffice to establish that the continuity of a function at point N does *not* necessarily imply that the graph of the function is "smooth" at $v = N$, for the point (N, L) in Fig. 6.2*b* is actually a "sharp" point and yet the function is continuous at that value of v.

When a function $q = g(v)$ is continuous at all values of v in the interval (a, b), it is said to be continuous in that interval. If the function is continuous at all points in a subset S of the domain (where the subset S may be the union of several disjoint intervals), it is said to be continuous in S. And, finally, if the function is continuous at all points in its domain, we say that it is continuous in its domain. Even in this latter case, however, the graph of the function may nevertheless show a discontinuity (a gap) at some value of v, say, at $v = 5$, if that value of v is *not* in its domain.

Again referring to Fig. 6.2, we see that in diagram c the function is *discontinuous* at N because a limit does not exist at that point, in violation of the second requirement of continuity. Nevertheless, the function does satisfy the requirements of continuity in the interval $(0, N)$ of the domain, as well as in the interval $[N, \infty)$. Diagram d obviously is also discontinuous at $v = N$. This time, discontinuity emanates from the fact that N is excluded from the domain, in violation of the first requirement of continuity.

On the basis of the graphs in Fig. 6.2, it appears that sharp points are consistent with continuity, as in diagram b, but that gaps are taboo, as in diagrams c and d. This is indeed the case. Roughly speaking, therefore, a function that is continuous in a particular interval is one whose graph can be drawn for the said interval without lifting the pencil or pen from the paper—a feat which is possible even if there are sharp points, but impossible when gaps occur.

Polynomial and Rational Functions

Let us now consider the continuity of certain frequently encountered functions. For any polynomial function, such as $q = g(v)$ in (6.11), we have found from (6.12) that $\lim_{v \to N} q$ exists and is equal to the value of the function at N. Since N is a point (any point) in the domain of the function, we can conclude that any polynomial function is continuous in its domain. This is a very useful piece of information, because polynomial functions will be encountered very often.

What about rational functions? Regarding continuity, there exists an interesting theorem (the continuity theorem) which states that the sum, difference, product, and quotient of any finite number of functions that are continuous in the domain are, respectively, also continuous in the domain. As a result, any rational function (a quotient of two polynomial functions) must also be continuous in its domain.

Example 1 The rational function

$$q = g(v) = \frac{4v^2}{v^2 + 1}$$

is defined for all finite real numbers; thus its domain consists of the interval $(-\infty, \infty)$. For any number N in the domain, the limit of q is (by the quotient limit theorem)

$$\lim_{v \to N} q = \frac{\lim_{v \to N} (4v^2)}{\lim_{v \to N} (v^2 + 1)} = \frac{4N^2}{N^2 + 1}$$

which is equal to $g(N)$. Thus the three requirements of continuity are all met at N. Moreover, we note that N can represent any point in the domain of this function; consequently, this function is continuous in its domain.

Example 2 The rational function

$$q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}$$

is not defined at $v = 2$ and at $v = -2$. Since those two values of v are not in the domain, the function is discontinuous at $v = -2$ and $v = 2$, despite the fact that a limit of q exists as $v \to -2$ or 2. Graphically, this function will display a gap at each of these two values of v. But for other values of v (those which *are* in the domain), this function is continuous.

Differentiability of a Function

The previous discussion has provided us with the tools for ascertaining whether any function has a limit as its independent variable approaches some specific value. Thus we can try to take the limit of any function $y = f(x)$ as x approaches some chosen value, say, x_0. However, we can also apply the "limit" concept at a different level and take the limit of the difference quotient of that function, $\Delta y / \Delta x$, as Δx approaches zero. The outcomes of limit-taking at these two different levels relate to two different, though related, properties of the function f.

Taking the limit of the function $y = f(x)$ itself, we can, in line with the discussion of the preceding subsection, examine whether the function f is *continuous* at $x = x_0$. The conditions for continuity are (1) $x = x_0$ must be in the domain of the function f, (2) y must have a limit as $x \to x_0$, and (3) the said limit must be equal to $f(x_0)$. When these are satisfied, we can write

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{[continuity condition]} \tag{6.13}$$

When the "limit" concept is applied to the difference quotient $\Delta y / \Delta x$ as $\Delta x \to 0$, on the other hand, we deal instead with the question of whether the function f is *differentiable* at $x = x_0$, i.e., whether the derivative dy/dx exists at

$x = x_0$, or whether $f'(x_0)$ exists. The term "differentiable" is used here because the process of obtaining the derivative dy/dx is known as *differentiation* (also called *derivation*). Since $f'(x_0)$ exists if and only if the limit of $\Delta y/\Delta x$ exists at $x = x_0$ as $\Delta x \to 0$, the symbolic expression of the differentiability of f is

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{6.14}$$

$$\equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad \text{[differentiability condition]}$$

These two properties, continuity and differentiability, are very intimately related to each other—the continuity of f is a *necessary* condition for its differentiability (although, as we shall see later, this condition is *not sufficient*). What this means is that, to be differentiable at $x = x_0$, the function must first pass the test of being continuous at $x = x_0$. To prove this, we shall demonstrate that, given a function $y = f(x)$, its continuity at $x = x_0$ follows from its differentiability at $x = x_0$, i.e., condition (6.13) follows from condition (6.14). Before doing this, however, let us simplify the notation somewhat by (1) replacing x_0 with the symbol N and (2) replacing $(x_0 + \Delta x)$ with the symbol x. The latter is justifiable because the postchange value of x can be any number (depending on the magnitude of the change) and hence is a variable denotable by x. The equivalence of the two notation systems is shown in Fig. 6.4, where the old notations appear (in brackets) alongside the new. Note that, with the notational change, Δx now becomes $(x - N)$, so that the expression "$\Delta x \to 0$" becomes

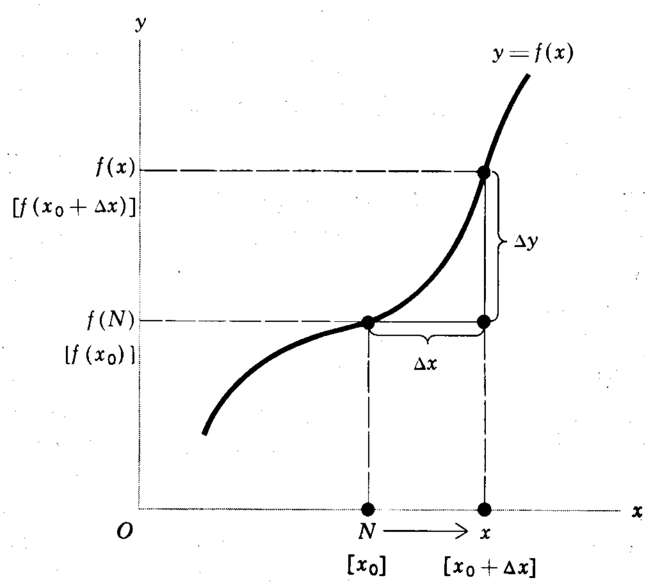

Figure 6.4

"$x \to N$," which is analogous to the expression $v \to N$ used before in connection with the function $q = g(v)$. Accordingly, (6.13) and (6.14) can now be rewritten, respectively, as

$$\lim_{x \to N} f(x) = f(N) \tag{6.13'}$$

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \tag{6.14'}$$

What we want to show is, therefore, that the continuity condition (6.13') follows from the differentiability condition (6.14'). First, since the notation $x \to N$ implies that $x \neq N$, so that $x - N$ is a nonzero number, it is permissible to write the following identity:

$$f(x) - f(N) \equiv \frac{f(x) - f(N)}{x - N}(x - N) \tag{6.15}$$

Taking the limit of each side of (6.15) as $x \to N$ yields the following results:

$$\begin{aligned} \text{Left side} &= \lim_{x \to N} f(x) - \lim_{x \to N} f(N) && \text{[difference limit theorem]} \\ &= \lim_{x \to N} f(x) - f(N) && [f(N) \text{ is a constant}] \\ \text{Right side} &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \lim_{x \to N}(x - N) && \text{[product limit theorem]} \\ &= f'(N)\left(\lim_{x \to N} x - \lim_{x \to N} N\right) && \text{[by (6.14') and difference limit theorem]} \\ &= f'(N)(N - N) = 0 \end{aligned}$$

Note that we could not have written these results, if condition (6.14') had not been granted, for if $f'(N)$ did not exist, then the right-side expression (and hence also the left-side expression) in (6.15) would not possess a limit. If $f'(N)$ does exist, however, the two sides will have limits as shown above. Moreover, when the left-side result and the right-side result are equated, we get $\lim_{x \to N} f(x) - f(N) = 0$, which is identical with (6.13'). Thus we have proved that continuity, as shown in (6.13'), follows from differentiability, as shown in (6.14'). In general, if a function is differentiable at every point in its domain, we may conclude that it must be continuous in its domain.

Although differentiability implies continuity, the converse is not true. That is, continuity is a *necessary*, but *not* a *sufficient*, condition for differentiability. To demonstrate this, we merely have to produce a counterexample. Let us consider the function

$$y = f(x) = |x - 2| + 1 \tag{6.16}$$

which is graphed in Fig. 6.5. As can be readily shown, this function is not differentiable, though continuous, when $x = 2$. That the function is continuous at $x = 2$ is easy to establish. First, $x = 2$ is in the domain of the function. Second, the limit of y exists as x tends to 2; to be specific, $\lim_{x\to 2^+} y = \lim_{x\to 2^-} y = 1$. Third, $f(2)$ is also found to be 1. Thus all three requirements of continuity are met. To show that the function f is *not* differentiable at $x = 2$, we must show that the limit of the difference quotient

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{|x-2| + 1 - 1}{x - 2} = \lim_{x\to 2} \frac{|x-2|}{x-2}$$

does *not* exist. This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit, x must exceed 2, according to the definition of absolute value in (6.8) we have $|x - 2| = x - 2$. Thus the right-side limit is

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$$

On the other hand, in considering the left-side limit, x must be less than 2; thus, according to (6.8), $|x - 2| = -(x - 2)$. Consequently, the left-side limit is

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = \lim_{x\to 2^-} (-1) = -1$$

which is different from the right-side limit. This shows that continuity does not guarantee differentiability. In sum, all differentiable functions are continuous, but not all continuous functions are differentiable.

In Fig. 6.5, the nondifferentiability of the function at $x = 2$ is manifest in the fact that the point (2, 1) has no tangent line defined, and hence no definite slope can be assigned to the point. Specifically, to the left of that point, the curve has a

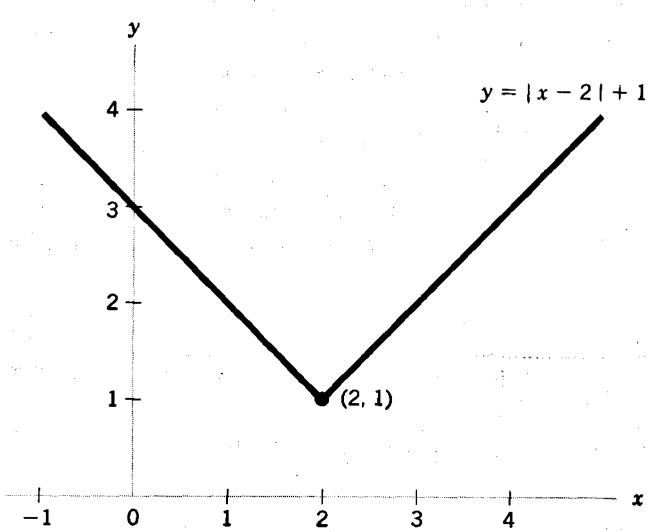

Figure 6.5

slope of -1, but to the right it has a slope of $+1$, and the slopes on the two sides display no tendency to approach a common magnitude at $x = 2$. The point (2, 1) is, of course, a special point; it is the only sharp point on the curve. At other points on the curve, the derivative is defined and the function is differentiable. More specifically, the function in (6.16) can be divided into two linear functions as follows:

Left part: $y = -(x-2)+1 = 3-x \qquad (x \le 2)$

Right part: $y = (x-2)+1 = x-1 \qquad (x > 2)$

The left part is differentiable in the interval $(-\infty, 2)$, and the right part is differentiable in the interval $(2, \infty)$ in the domain.

In general, differentiability is a more restrictive condition than continuity, because it requires something beyond continuity. Continuity at a point only rules out the presence of a gap, whereas differentiability rules out "sharpness" as well. Therefore, differentiability calls for "smoothness" of the function (curve) as well as its continuity. Most of the *specific* functions employed in economics have the property that they are differentiable everywhere. When *general* functions are used, moreover, they are often assumed to be everywhere differentiable, as we shall do in the subsequent discussion.

EXERCISE 6.7

1 A function $y = f(x)$ is discontinuous at $x = x_0$ when *any* of the three requirements for continuity is violated at $x = x_0$. Construct three graphs to illustrate the violation of each of those requirements.

2 Taking the set of all finite real numbers as the domain of the function $q = g(v) = v^2 - 7v - 3$:

(a) Find the limit of q as v tends to N (a finite real number).

(b) Check whether this limit is equal to $g(N)$.

(c) Check whether the function is continuous at N and continuous in its domain.

3 Given the function $q = g(v) = \dfrac{v+2}{v^2+2}$:

(a) Use the limit theorems to find $\lim_{v \to N} q$, N being a finite real number.

(b) Check whether this limit is equal to $g(N)$.

(c) Check the continuity of the function $g(v)$ at N and in its domain $(-\infty, \infty)$.

4 Given $y = f(x) = \dfrac{x^2 + x - 20}{x - 4}$:

(a) Is it possible to apply the quotient limit theorem to find the limit of this function as $x \to 4$?

(b) Is this function continuous at $x = 4$? Why?

(c) Find a function which, for $x \neq 4$, is equivalent to the above function, and obtain from the equivalent function the limit of y as $x \to 4$.

5 In the rational function in Example 2, the numerator is evenly divisible by the denominator, and the quotient is $v + 1$. Can we for that reason replace that function outright by $q = v + 1$? Why or why not?

6 On the basis of the graphs of the six functions in Fig. 2.8, would you conclude that each such function is differentiable at every point in its domain? Explain.