3 Use the chain rule to find dy/dx for the following:
(a) $y = (3x^2 - 13)^3$ (b) $y = (8x^3 - 5)^9$ (c) $y = (ax + b)^4$

4 Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?

5 Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.

6 Are the following functions monotonic?
(a) $y = -x^6 + 5 \quad (x > 0)$ (b) $y = 4x^5 + x^3 + 3x$
For each monotonic function, find dx/dy by the inverse-function rule.

7.4 PARTIAL DIFFERENTIATION

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while $x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1} \tag{7.13}$$

If we take the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1, to indicate that all the other independent variables in the function are held constant when taking this particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called *partial differentiation*.

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in dy/dx), we employ the symbol ∂, which is a variant of the Greek δ (lower case delta). Thus we shall now write $\partial y/\partial x_i$, which is read: "the partial derivative of y

with respect to x_i." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i} y$; in that case, its $\partial/\partial x_i$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable x_i. Since the function involved here is denoted in (7.12) by f, it is also permissible to write $\partial f/\partial x_i$.

Is there also a partial-derivative counterpart for the symbol $f'(x)$ that we used before? The answer is yes. Instead of f', however, we now use f_1, f_2, etc., where the subscript indicates which independent variable (alone) is being allowed to vary. If the function in (7.12) happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, then the partial derivatives may be denoted by f_u, f_v, and f_w rather than f_1, f_2, and f_3.

In line with these notations, and on the basis of (7.12) and (7.13), we can now define

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

as the first in the set of n partial derivatives of the function f.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold $(n - 1)$ independent variables *constant* while allowing *one* variable to vary. Inasmuch as we have learned how to handle *constants* in differentiation, the actual differentiation should pose little problem.

Example 1 Given $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find the partial derivatives. When finding $\partial y/\partial x_1$ (or f_1), we must bear in mind that x_2 is to be treated as a constant during differentiation. As such, x_2 will drop out in the process if it is an *additive* constant (such as the term $4x_2^2$) but will be retained if it is a *multiplicative* constant (such as in the term x_1x_2). Thus we have

$$\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$$

Similarly, by treating x_1 as a constant, we find that

$$\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables x_1 and x_2. That is, we may write them as two derived functions

$$f_1 = f_1(x_1, x_2) \qquad \text{and} \qquad f_2 = f_2(x_1, x_2)$$

For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f, for example, the partial derivatives will take the following specific values:

$$f_1(1, 3) = 6(1) + 3 = 9 \qquad \text{and} \qquad f_2(1, 3) = 1 + 8(3) = 25$$

Example 2 Given $y = f(u, v) = (u + 4)(3u + 2v)$, the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = (u + 4)(3) + 1(3u + 2v) = 2(3u + v + 6)$$

Similarly, by holding u constant, we find that

$$f_v = (u + 4)(2) + 0(3u + 2v) = 2(u + 4)$$

When $u = 2$ and $v = 1$, these derivatives will take the following values:

$$f_u(2, 1) = 2(13) = 26 \qquad \text{and} \qquad f_v(2, 1) = 2(6) = 12$$

Example 3 Given $y = (3u - 2v)/(u^2 + 3v)$, the partial derivatives can be found by use of the quotient rule:

$$\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - 2u(3u - 2v)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - 3(3u - 2v)}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}$$

Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instantaneous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

Let us consider a production function $Q = Q(K, L)$, where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (7.12), with $n = 2$. We can therefore define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). The partial derivative Q_K relates to the rates of change in output with respect to infinitesimal changes in capital, while labor input is held constant. Thus Q_K symbolizes the marginal-physical-product-of-capital (MPP_K) function. Similarly, the partial derivative Q_L is the mathematical representation of the MPP_L function.

Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, such as is shown in Fig. 7.4. The variable Q is plotted vertically, so that for any point (K, L) in the base plane (KL plane), the height of the surface will indicate the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the rectangle OK_0BL_0. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level K_0 and consider only variations in the input L. By setting $K = K_0$, all points in our (curtailed) domain become irrelevant except those on the line segment K_0B. By the same token, only the curve K_0CDA (a cross section of the production surface) will be germane to the present discussion. This curve represents a total-physical-product-of-labor (TPP_L)

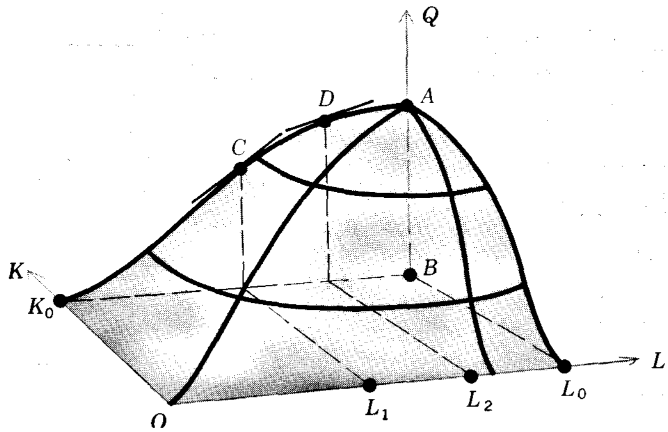

Figure 7.4

curve for a fixed amount of capital $K = K_0$; thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. Once again, we note that the slope of a total (TPP_L) curve is its corresponding marginal ($\text{MPP}_L \equiv Q_L$) curve.

It was mentioned earlier that a partial derivative is a function of all the independent variables of the primitive function. That Q_L is a function of L is immediately obvious from the K_0CDA curve itself. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C; but when $L = L_2$, the relevant slope is the one at point D. Why is Q_L also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there will result a different TPP_L curve (a different cross section of the production surface), with inevitable repercussions on the derivative Q_L. Hence Q_L is also a function of K.

An analogous interpretation can be given to the partial derivative Q_K. If the labor input is held constant instead of K (say, at the level of L_0), the line segment L_0B will be the relevant subset of the domain, and the curve L_0A will indicate the relevant subset of the production surface. The partial derivative Q_K can then be interpreted as the slope of the curve L_0A—bearing in mind that the K axis extends from southeast to northwest in Fig. 7.4. It should be noted that Q_K is again a function of both the variables L and K.

EXERCISE 7.4

1 Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:

(*a*) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$ (*c*) $y = (2x_1 + 3)(x_2 - 2)$

(*b*) $y = 7x_1 + 5x_1x_2^2 - 9x_2^3$ (*d*) $y = (4x_1 + 3)/(x_2 - 2)$

2 Find f_x and f_y from the following:

(a) $f(x, y) = x^2 + 5xy - y^3$ (c) $f(x, y) = \frac{2x - 3y}{x + y}$

(b) $f(x, y) = (x^2 - 3y)(x - 2)$ (d) $f(x, y) = \frac{x^2 - 1}{xy}$

3 From the answers to the preceding problem, find $f_x(1, 2)$—the value of the partial derivative f_x when $x = 1$ and $y = 2$—for each function.

4 Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MPP_K a function of K alone, or of both K and L? What about MPP_L?

5 If the utility function of an individual takes the form

$$U = U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^3$$

where U is total utility, and x_1 and x_2 are the quantities of two commodities consumed:

(a) Find the marginal-utility function of each of the two commodities.

(b) Find the value of the marginal utility of the first commodity when 3 units of each commodity are consumed.

7.5 APPLICATIONS TO COMPARATIVE-STATIC ANALYSIS

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations:

$$Q = a - bP \qquad (a, b > 0) \qquad \text{[demand]}$$

$$Q = -c + dP \qquad (c, d > 0) \qquad \text{[supply]}$$

with solutions

$$\bar{P} = \frac{a + c}{b + d} \tag{7.14}$$

$$\bar{Q} = \frac{ad - bc}{b + d} \tag{7.15}$$

These solutions will be referred to as being in the *reduced form*: the two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of $\bar{P}$, one has only to differentiate (7.14) partially with respect to each of the parameters. If the *sign* of a partial derivative, say, $\partial\bar{P}/\partial a$, can be determined