PART FOUR

OPTIMIZATION PROBLEMS

CHAPTER NINE

OPTIMIZATION: A SPECIAL VARIETY OF EQUILIBRIUM ANALYSIS

When we first introduced the term equilibrium in Chap. 3, we made a broad distinction between goal and nongoal equilibrium. In the latter type, exemplified by our study of market and national-income models, the interplay of certain opposing forces in the model—e.g., the forces of demand and supply in the market models and the forces of leakages and injections in the income models—dictates an equilibrium state, if any, in which these opposing forces are just balanced against each other, thus obviating any further tendency to change. The attainment of this type of equilibrium is the outcome of the impersonal balancing of these forces and does not require the conscious effort on the part of anyone to accomplish a specified goal. True, the consuming households behind the forces of demand and the firms behind the forces of supply are each striving for an optimal position under the given circumstances, but as far as the market itself is concerned, no one is aiming at any particular equilibrium price or equilibrium quantity (unless, of course, the government happens to be trying to peg the price). Similarly, in national-income determination, the impersonal balancing of leakages and injections is what brings about an equilibrium state, and no conscious effort at reaching any particular goal (such as an attempt to alter an undesirable income level by means of monetary or fiscal policies) needs to be involved at all.

In the present part of the book, however, our attention will be turned to the study of *goal equilibrium*, in which the equilibrium state is defined as the optimum position for a given economic unit (a household, a business firm, or even an entire economy) and in which the said economic unit will be deliberately striving for attainment of that equilibrium. As a result, in this context—but only in this context—our earlier warning that equilibrium does not imply desirability will become irrelevant and immaterial. In this part of the book, our primary focus will be on the classical techniques for locating optimum positions—those using differential calculus. More modern developments, known as mathematical programming, will be discussed later.

9.1 OPTIMUM VALUES AND EXTREME VALUES

Economics is by and large a science of choice. When an economic project is to be carried out, such as the production of a specified level of output, there are normally a number of alternative ways of accomplishing it. One (or more) of these alternatives will, however, be more desirable than others from the standpoint of some criterion, and it is the essence of the optimization problem to choose, on the basis of that specified criterion, the best alternative available.

The most common criterion of choice among alternatives in economics is the goal of *maximizing* something (such as maximizing a firm's profit, a consumer's utility, or the rate of growth of a firm or of a country's economy) or of *minimizing* something (such as minimizing the cost of producing a given output). Economically, we may categorize such maximization and minimization problems under the general heading of *optimization*, meaning "the quest for the best." From a purely mathematical point of view, however, the terms "maximum" and "minimum" do not carry with them any connotation of optimality. Therefore, the collective term for maximum and minimum, as mathematical concepts, is the more matter-of-fact designation *extremum*, meaning an extreme value.

In formulating an optimization problem, the first order of business is to delineate an *objective function* in which the dependent variable represents the object of maximization or minimization and in which the set of independent variables indicates the objects whose magnitudes the economic unit in question can pick and choose, with a view to optimizing. We shall therefore refer to the independent variables as *choice variables*.* The essence of the optimization process is simply to find the set of values of the choice variables that will yield the desired extremum of the objective function.

For example, a business firm may seek to maximize profit π, that is, to maximize the difference between total revenue R and total cost C. Since, within the framework of a given state of technology and a given market demand for the firm's product, R and C are both functions of the output level Q, it follows that π

* They can also be called *decision variables*, or *policy variables*.

is also expressible as a function of Q:

$$\pi(Q) = R(Q) - C(Q)$$

This equation constitutes the relevant objective function, with π as the object of maximization and Q as the (only) choice variable. The optimization problem is then that of choosing the level of Q such that π will be a maximum. Note that the *optimal* level of π is by definition its *maximal* level, but the optimal level of the choice variable Q is itself not required to be either a maximum or a minimum.

To cast the problem into a more general mold for further discussion (though still confining ourselves to objective functions of one variable only), let us consider the general function

$$y = f(x)$$

and attempt to develop a procedure for finding the level of x that will maximize or minimize the value of y. It will be assumed in this discussion that the function f is continuously differentiable.

9.2 RELATIVE MAXIMUM AND MINIMUM: FIRST-DERIVATIVE TEST

Since the objective function $y = f(x)$ is stated in the general form, there is no restriction as to whether it is linear or nonlinear or whether it is monotonic or contains both increasing and decreasing parts. From among the many possible types of function compatible with the above objective-function form, we have selected three specific cases to be depicted in Fig. 9.1. Simple as they may be, the graphs in Fig. 9.1 should give us valuable insight into the problem of locating the maximum or minimum value of the function $y = f(x)$.

Relative versus Absolute Extremum

If the objective function is a constant function, as in Fig. 9.1a, all values of the choice variable x will result in the same value of y, and the height of each point

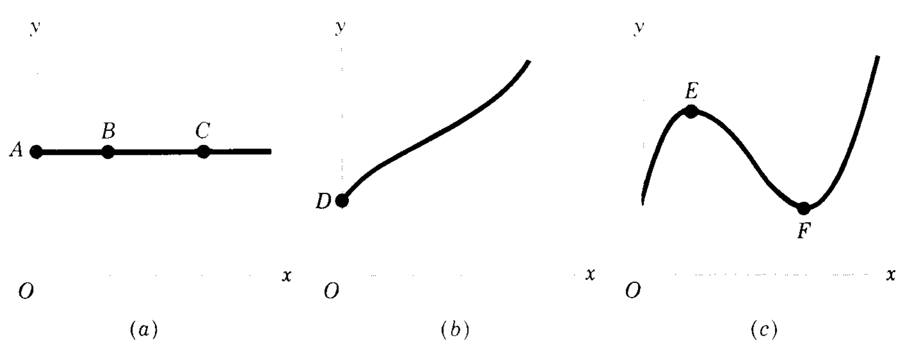

Figure 9.1

on the graph of the function (such as A or B or C) may be considered a maximum or, for that matter, a minimum—or, indeed, neither. In this case, there is in effect no significant choice to be made regarding the value of x for the maximization or minimization of y.

In Fig. 9.1*b*, the function is monotonically increasing, and there is no finite maximum if the set of nonnegative real numbers is taken to be its domain. However, we may consider the end point D on the left (the y intercept) as representing a minimum; in fact, it is in this case the *absolute* (or *global*) minimum in the range of the function.

The points E and F in Fig. 9.1*c*, on the other hand, are examples of a *relative* (or *local*) extremum, in the sense that each of these points represents an extremum in the immediate neighborhood of the point only. The fact that point F is a relative minimum is, of course, no guarantee that it is also the global minimum of the function, although this may happen to be the case. Similarly, a relative maximum point such as E may or may not be a global maximum. Note also that a function can very well have several relative extrema, some of which may be maxima while others are minima.

In most economic problems that we shall be dealing with, our primary, if not exclusive, concern will be with extreme values other than end-point values, for with most such problems the domain of the objective function is restricted to be the set of nonnegative numbers, and thus an end point (on the left) will represent the zero level of the choice variable, which is often of no practical interest. Actually, the type of function most frequently encountered in economic analysis is that shown in Fig. 9.1*c*, or some variant thereof which contains only a single bend in the curve. We shall therefore continue our discussion mainly with reference to the search for *relative* extrema such as points E and F. This will, however, by no means foreclose the knowledge of an absolute maximum if we want it, because an absolute maximum must be either a relative maximum or one of the end points of the function. Thus if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the absolute maximum. The absolute minimum of a function can be found analogously. Hereafter, the extreme values considered will be *relative* or *local* ones, unless indicated otherwise.

First-Derivative Test

As a matter of terminology, from now on we shall refer to the derivative of a function alternatively as its *first* derivative (short for *first-order* derivative). The reason for this will become apparent shortly.

Given a function $y = f(x)$, the first derivative $f'(x)$ plays a major role in our search for its extreme values. This is due to the fact that, if a relative extremum of the function occurs at $x = x_0$, then either (1) we have $f'(x_0) = 0$, or (2) $f'(x_0)$ does not exist. The second eventuality is illustrated in Fig. 9.2*a*, where both points A and B depict relative extreme values of y, and yet no derivative is defined at either of these sharp points. Since in the present discussion we are assuming that $y = f(x)$ is continuous and possesses a continuous derivative, however, we are in effect ruling out sharp points. For smooth functions, relative extreme values can occur only where the first derivative has a zero value. This is illustrated by points C and D in Fig. 9.2b, both of which represent extreme values, and both of which are characterized by a zero slope—$f'(x_1) = 0$ and $f'(x_2) = 0$. It is also easy to see that when the slope is nonzero we cannot possibly have a relative minimum (the bottom of a valley) or a relative maximum (the peak of a hill). For this reason, we can, in the context of smooth functions, take the condition $f'(x) = 0$ as a *necessary* condition for a relative extremum (either maximum or minimum).

We must add, however, that a zero slope, while *necessary*, is *not sufficient* to establish a relative extremum. An example of the case where a zero slope is not associated with an extremum will be presented shortly. By appending a certain proviso to the zero-slope condition, however, we can obtain a decisive test for a relative extremum. This may be stated as follows:

First-derivative test for relative extremum If the first derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.
b. A relative *minimum* if $f'(x)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right.
c. Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of point x_0.

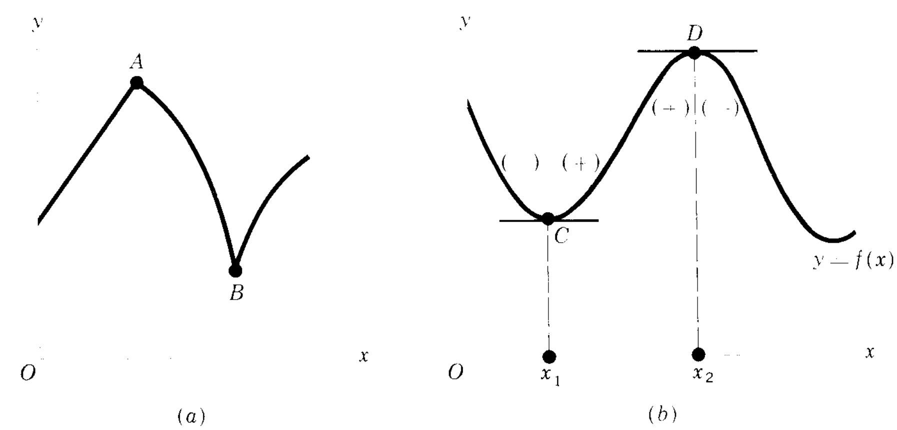

Figure 9.2

Let us call the value x_0 a *critical value* of x if $f'(x_0) = 0$, and refer to $f(x_0)$ as a *stationary value* of y (or of the function f). The point with coordinates x_0 and $f(x_0)$ can, accordingly, be called a *stationary point*. (The rationale for the word "stationary" should be self-evident—wherever the slope is zero, the point in question is never situated on an upward or downward incline, but is rather at a standstill position.) Then, graphically, the first possibility listed in this test will establish the stationary point as the peak of a hill, such as point D in Fig. 9.2b, whereas the second possibility will establish the stationary point as the bottom of a valley, such as point C in the same diagram. Note, however, that in view of the existence of a third possibility, yet to be discussed, we are unable to regard the condition $f'(x) = 0$ as a *sufficient condition* for a relative extremum. But we now see that, *if* the necessary condition $f'(x) = 0$ is satisfied, *then* the change-of-derivative-sign proviso can serve as a *sufficient condition* for a relative maximum or minimum, depending on the direction of the sign change.

Let us now explain the third possibility. In Fig. 9.3a, the function f is shown to attain a zero slope at point J (when $x = j$). Even though $f'(j)$ is zero—which makes $f(j)$ a stationary value—the derivative does not change its sign from one side of $x = j$ to the other; therefore, according to the test above, point J gives neither a maximum nor a minimum, as is duly confirmed by the graph of the function. Rather, it exemplifies what is known as an *inflection point*.

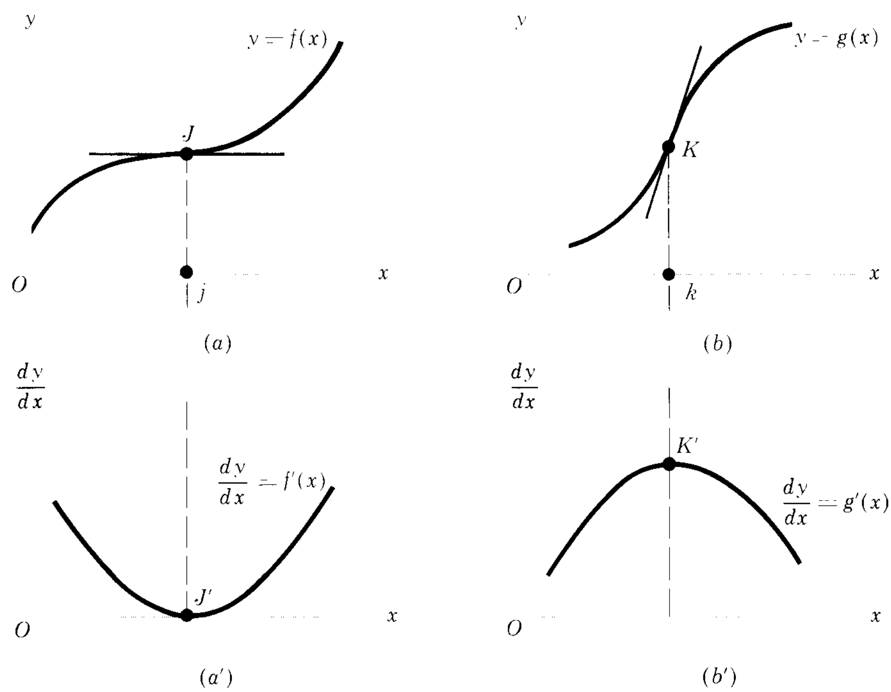

Figure 9.3

The characteristic feature of an inflection point is that, at that point, the derivative (as against the primitive) function reaches an extreme value. Since this extreme value can be either a maximum or a minimum, we have two types of inflection points. In Fig. 9.3a', where we have plotted the derivative $f'(x)$, we see that its value is zero when $x = j$ (see point J') but is positive on both sides of point J'; this makes J' a *minimum* point of the derivative function $f'(x)$.

The other type of inflection point is portrayed in Fig 9.3b, where the slope of the function $g(x)$ increases till the point k is reached and decreases thereafter. Consequently, the graph of the derivative function $g'(x)$ will assume the shape shown in diagram b', where point K' gives a *maximum* value of the derivative function $g'(x)$.*

To sum up: A relative extremum must be a stationary value, but a stationary value may be associated with either a relative extremum or an inflection point. To find the relative maximum or minimum of a given function, therefore, the procedure should be first to find the stationary values of the function where $f'(x) = 0$ and then to apply the first-derivative test to determine whether each of the stationary values is a relative maximum, a relative minimum, or neither.

Example 1 Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8$$

First, we find the derivative function to be

$$f'(x) = 3x^2 - 24x + 36$$

To get the critical values, i.e., the values of x satisfying the condition $f'(x) = 0$, we set the quadratic derivative function equal to zero and get the quadratic equation

$$3x^2 - 24x + 36 = 0$$

By factoring the polynomial or by applying the quadratic formula, we then obtain the following pair of roots (solutions):

$$\bar{x}_1 = 2 \quad [\text{at which we have } f'(2) = 0 \text{ and } f(2) = 40]$$

$$\bar{x}_2 = 6 \quad [\text{at which we have } f'(6) = 0 \text{ and } f(6) = 8]$$

Since $f'(2) = f'(6) = 0$, these two values of x are the critical values we desire.

It is easy to verify that $f'(x) > 0$ for $x < 2$, and $f'(x) < 0$ for $x > 2$, in the immediate neighborhood of $x = 2$; thus, the corresponding value of the function $f(2) = 40$ is established as a relative maximum. Similarly, since $f'(x) < 0$ for $x < 6$, and $f'(x) > 0$ for $x > 6$, in the immediate neighborhood of $x = 6$, the value of the function $f(6) = 8$ must be a relaive minimum.

* Note that a zero derivative value, while a necessary condition for a relative extremum, is *not* required for an inflection point; for the derivative $g'(x)$ has a positive value at $x = k$, and yet point K is an inflection point.

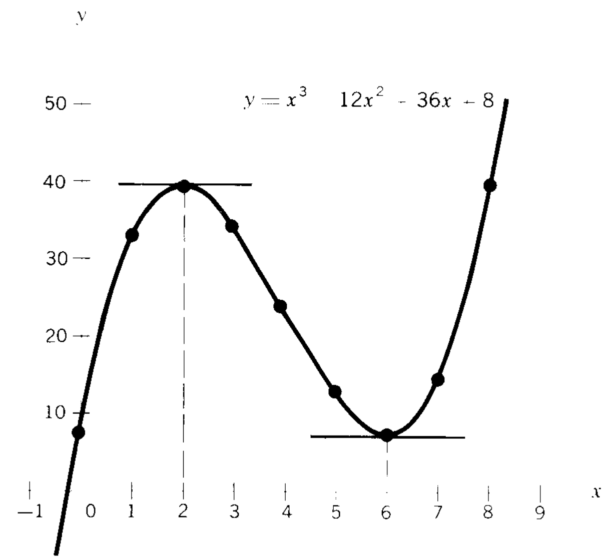

Figure 9.4

The graph of the function of this example is shown in Fig. 9.4. Such a graph may be used to verify the location of extreme values obtained through use of the first-derivative test. But, in reality, in most cases "helpfulness" flows in the opposite direction—the mathematically derived extreme values will help in plotting the graph. The accurate plotting of a graph ideally requires knowledge of the value of the function at every point in the domain; but as a matter of actual practice, only a few points in the domain are selected for purposes of plotting, and the rest of the points typically are filled in by interpolation. The pitfall of this practice is that, unless we hit upon the stationary point(s) by coincidence, we shall miss the exact location of the turning point(s) in the curve. Now, with the first-derivative test at our disposal, it becomes possible to determine these turning points precisely.

Example 2 Find the relative extremum of the average-cost function

$$AC = f(Q) = Q^2 - 5Q + 8$$

The derivative here is $f'(Q) = 2Q - 5$, a linear function. Setting $f'(Q)$ equal to zero, we get the linear equation $2Q - 5 = 0$, which has the single root $\bar{Q} = 2.5$. This is the only critical value in this case. To apply the first-derivative test, let us find the values of the derivative at, say, $Q = 2.4$ and $Q = 2.6$, respectively. Since $f'(2.4) = -0.2 < 0$ whereas $f'(2.6) = 0.2 > 0$, we can conclude that the stationary value $AC = f(2.5) = 1.75$ represents a relative minimum. The graph of the function of this example is actually a U-shaped curve, so that the relative minimum already found will also be the absolute minimum. Our knowledge of the exact location of this point should be of great help in plotting the AC curve.

EXERCISE 9.2

1 Find the stationary values of the following (check whether relative maxima or minima or inflection points), assuming the domain to be the set of all real numbers:

(*a*) $y = -2x^2 + 4x + 9$ (*c*) $y = x^2 + 3$

(*b*) $y = 5x^2 + x$ (*d*) $y = 3x^2 - 6x + 2$

2 Find the stationary values of the following (check whether relative maxima or minima or inflection points), assuming the domain to be the interval $[0, \infty)$:

(*a*) $y = x^3 - 3x + 5$

(*b*) $y = \frac{1}{3}x^3 - x^2 + x + 10$

(*c*) $y = -x^3 + 4.5x^2 - 6x + 6$

3 Show that the function $y = x + 1/x$ (with $x \neq 0$) has two relative extrema, one a maximum and the other a minimum. Is the "minimum" larger or smaller than the "maximum"? How is this paradoxical result possible?

4 Let $T = \phi(x)$ be a *total* function (e.g., total product or total cost):

(*a*) Write out the expressions for the *marginal* function M and the *average* function A.

(*b*) Show that, when A reaches a relative extremum, M and A must have the same value.

(*c*) What general principle does this suggest for the drawing of a marginal curve and an average curve in the same diagram?

(*d*) What can you conclude about the elasticity of the total function T at the point where A reaches an extreme value?

9.3 SECOND AND HIGHER DERIVATIVES

Hitherto we have considered only the first derivative $f'(x)$ of a function $y = f(x)$; now let us introduce the concept of *second derivative* (short for *second-order derivative*), and derivatives of even higher orders. These will enable us to develop alternative criteria for locating the relative extrema of a function.

Derivative of a Derivative

Since the first derivative $f'(x)$ is itself a function of x, it, too, should be differentiable with respect to x, provided that it is continuous and smooth. The result of this differentiation, known as the second derivative of the function f, is denoted by

$f''(x)$ where the double prime indicates that $f(x)$ has been differentiated with respect to x twice, and where the expression (x) following the double prime suggests that the second derivative is again a function of x

or

$\frac{d^2y}{dx^2}$ where the notation stems from the consideration that the second derivative means, in fact, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$; hence the d^2 in the numerator and dx^2 in the denominator of this symbol

If the second derivative $f''(x)$ exists for all x values in the domain, the function $f(x)$ is said to be *twice differentiable*; if, in addition, $f''(x)$ is continuous, the function $f(x)$ is said to be *twice continuously differentiable*.*

As a function of x the second derivative can be differentiated with respect to x again to produce a *third* derivative, which in turn can be the source of a *fourth* derivative, and so on ad infinitum, as long as the differentiability condition is met. These higher-order derivatives are symbolized along the same line as the second derivative:

$$f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x) \qquad \text{[with superscripts enclosed in ( )]}$$

or

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}$$

The last of these can also be written as $\frac{d^n}{dx^n}y$, where the $\frac{d^n}{dx^n}$ part serves as an operator symbol instructing us to take the nth derivative of (some function) with respect to x.

Almost all the *specific* functions we shall be working with possess continuous derivatives up to any order we desire; i.e., they are continuously differentiable any number of times. Whenever a *general* function is used, such as $f(x)$, we always assume that it has derivatives up to any order we need.

Example 1 Find the first through the fifth derivatives of the function

$$y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$$

The desired derivatives are as follows:

$$f'(x) = 16x^3 - 3x^2 + 34x + 3$$
$$f''(x) = 48x^2 - 6x + 34$$
$$f'''(x) = 96x - 6$$
$$f^{(4)}(x) = 96$$
$$f^{(5)}(x) = 0$$

* The following notations are often used to denote continuity and differentiability of a function:

$f \in C^{(0)}$ or $f \in C$:	f is a continuous function
$f \in C^{(1)}$ or $f \in C'$:	f is continuously differentiable
$f \in C^{(2)}$:	f is twice continuously differentiable

The symbol $C^{(n)}$ denotes the set of all functions that possess nth-order derivatives which are continuous in the domain.

In this particular (polynomial-function) example, each successive derivative emerges as a simpler expression than the one before, until we reach a fifth derivative, which is identically zero. This is not generally true, however, of all types of function, as the next example will show. It should be stressed here that the statement "the fifth derivative is zero" is not the same as the statement "the fifth derivative does not exist," which describes an altogether different situation. Note, also, that $f^{(5)}(x) = 0$ (zero at all values of x) is not the same as $f^{(5)}(x_0) = 0$ (zero at x_0 only).

Example 2 Find the first four derivatives of the rational function

$$y = g(x) = \frac{x}{1+x} \qquad (x \neq -1)$$

These derivatives can be found either by use of the quotient rule, or, after rewriting the function as $y = x(1+x)^{-1}$, by the product rule:

$$\left.\begin{aligned} g'(x) &= (1+x)^{-2} \\ g''(x) &= -2(1+x)^{-3} \\ g'''(x) &= 6(1+x)^{-4} \\ g^{(4)}(x) &= -24(1+x)^{-5} \end{aligned}\right\} \qquad (x \neq -1)$$

In this case, repeated derivation evidently does not tend to simplify the subsequent derivative expressions.

Note that, like the primitive function $g(x)$, all the successive derivatives obtained are themselves functions of x. Given specific values of x, these derivative functions will then take specific values. When $x = 2$, for instance, the second derivative in Example 2 can be evaluated as

$$g''(2) = -2(3)^{-3} = \frac{-2}{27}$$

and similarly for other values of x. It is of the utmost importance to realize that to evaluate this second derivative $g''(x)$ at $x = 2$, as we did, we must first obtain $g''(x)$ from $g'(x)$ and then substitute $x = 2$ into the equation for $g''(x)$. It is *incorrect* to substitute $x = 2$ into $g(x)$ or $g'(x)$ *prior* to the differentiation process leading to $g''(x)$.

Interpretation of the Second Derivative

The derivative function $f'(x)$ measures the rate of change of the function f. By the same token, the second-derivative function f'' is the measure of the rate of change of the first derivative f'; in other words, the second derivative measures the *rate of change* of the *rate of change* of the original function f. To put it differently, with a given infinitesimal increase in the independent variable x from a point $x = x_0$,

$$\left.\begin{matrix} f'(x_0) > 0 \\ f'(x_0) < 0 \end{matrix}\right\} \text{ means that the } \textit{value of the function} \text{ tends to } \left\{\begin{matrix} \text{increase} \\ \text{decrease} \end{matrix}\right.$$

whereas, with regard to the second derivative,

$$\left.\begin{matrix} f''(x_0) > 0 \\ f''(x_0) < 0 \end{matrix}\right\} \text{ means that the } \textit{slope of the curve} \text{ tends to } \left\{\begin{matrix} \text{increase} \\ \text{decrease} \end{matrix}\right.$$

Thus a positive first derivative coupled with a positive second derivative at $x = x_0$ implies that the slope of the curve at that point is *positive and increasing*—the value of the function is increasing at an increasing rate. Likewise, a positive first derivative with a negative second derivative indicates that the slope of the curve is *positive but decreasing*—the value of the function is increasing at a decreasing rate. The case of a negative first derivative can be interpreted analogously, but a warning should accompany this case: When $f'(x_0) < 0$ and $f''(x_0) > 0$, the slope of the curve is *negative and increasing*, but this does *not* mean that the slope is changing, say, from (-10) to (-11); on the contrary, the change should be from (-11), a smaller number, to (-10), a larger number. In other words, the negative slope must tend to be *less* steep as x increases. Lastly, when $f'(x_0) < 0$ and $f''(x_0) < 0$, the slope of the curve must be *negative and decreasing*. This refers to a negative slope that tends to become *steeper* as x increases.

Since we have been talking about slopes, it may be useful to continue the discussion with a graphical illustration. In Fig. 9.5 we have marked out six points (A, B, C, D, E, and F) on the two parabolas shown; each of these points illustrates a different combination of first- and second-derivative signs, as follows:

If at	the derivative signs are		we can illustrate it by
$x = x_1$	$f'(x_1) > 0$	$f''(x_1) < 0$	point A
$x = x_2$	$f'(x_2) = 0$	$f''(x_2) < 0$	point B
$x = x_3$	$f'(x_3) < 0$	$f''(x_3) < 0$	point C
$x = x_4$	$g'(x_4) < 0$	$g''(x_4) > 0$	point D
$x = x_5$	$g'(x_5) = 0$	$g''(x_5) > 0$	point E
$x = x_6$	$g'(x_6) > 0$	$g''(x_6) > 0$	point F

From this, we see that a *negative* second derivative (the first three cases) is consistently reflected in an inverse U-shaped curve, or a portion thereof, because the curve in question is required to have a smaller and smaller slope as x increases. In contrast, a *positive* second derivative (the last three cases) consistently points to a U-shaped curve, or a portion thereof, since the curve in question must display a larger and larger slope as x increases. Viewing the two curves in Fig. 9.5 from the standpoint of the horizontal axis, we find the one in diagram a to be concave throughout, whereas the one in diagram b is convex throughout. Since concavity and convexity are descriptions of how the curve "bends," we may now expect the second derivative of a function to inform us about the *curvature* of its graph, just as the first derivative tells us about its *slope*.

Although the words "concave" and "convex" adequately convey the differing curvature of the two curves in Fig. 9.5, writers today would more specifically label them as *strictly concave* and *strictly convex*, respectively. In line with this terminol-

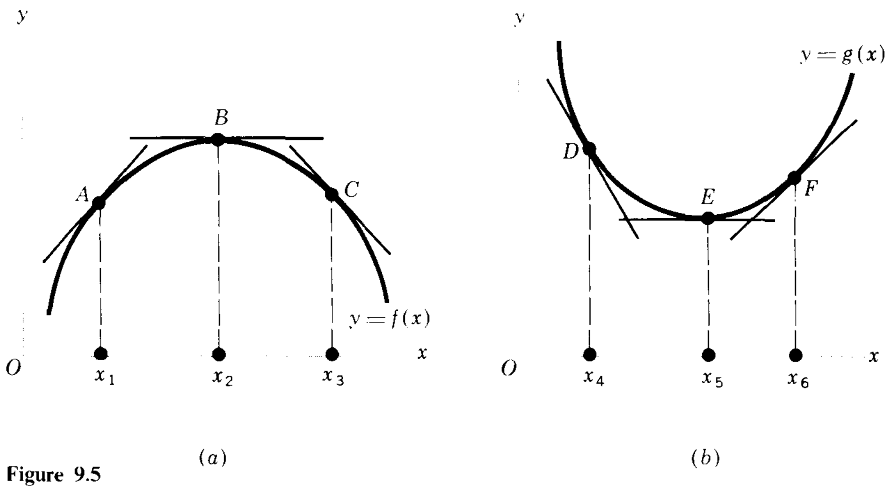

Figure 9.5

ogy, a function whose graph is strictly concave (strictly convex) is called a *strictly concave (strictly convex) function*. The precise geometric characterization of a strictly concave function is as follows. If we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely *below* the curve, except at points M and N. The characterization of a strictly convex function can be obtained by substituting the word "above" for the word "below" in the last statement. Try this out in Fig. 9.5. If the characterizing condition is relaxed somewhat, so that the line segment MN is allowed to lie *either* below the curve, *or* along (coinciding with) the curve, then we will be describing instead a *concave function*, without the adverb "strictly." Similarly, if the line segment MN *either* lies above, *or* lies along the curve, then the function is *convex*, again without the adverb "strictly." Note that, since the line segment MN may coincide with a (nonstrictly) concave or convex curve, the latter may very well contain a linear segment. In contrast, a *strictly* concave or convex curve can never contain a linear segment anywhere. It follows that while a strictly concave (convex) function is automatically a concave (convex) function, the converse is not true.*

From our earlier discussion of the second derivative, we may now infer that if the second derivative $f''(x)$ is negative for all x, then the primitive function $f(x)$ must be a strictly concave function. Similarly, $f(x)$ must be strictly convex, if $f''(x)$ is positive for all x. Despite this, it is *not* valid to reverse the above inference and say that, if $f(x)$ is strictly concave (strictly convex), then $f''(x)$ must be negative (positive) for all x. This is because, in certain exceptional cases, the second derivative may have a *zero* value at a stationary point on such a curve. An example of this can be found in the function $y = f(x) = x^4$, which plots as a strictly convex curve, but whose derivatives

$$f'(x) = 4x^3 \qquad f''(x) = 12x^2$$

* We shall discuss these concepts further in Sec. 11.5 below.

indicate that, at the stationary point where $x = 0$, the value of the second derivative is $f''(0) = 0$. Note, however, that at any other point, with $x \neq 0$, the second derivative of this function does have the (expected) positive sign. Aside from the possibility of a zero value at a stationary point, therefore, the second derivative of a strictly concave or convex function may be expected in general to adhere to a single algebraic sign.

For other types of function, the second derivative may take both positive and negative values, depending on the value of x. In Fig. 9.3*a* and *b*, for instance, both $f(x)$ and $g(x)$ undergo a sign change in the second derivative at their respective inflection points J and K. According to Fig. 9.3*a*′, the slope of $f'(x)$—that is, the value of $f''(x)$—changes from negative to positive at $x = j$; the exact opposite occurs with the slope of $g'(x)$—that is, the value of $g''(x)$—on the basis of Fig. 9.3*b*′. Translated into curvature terms, this means that the graph of $f(x)$ turns from concave to convex at point J, whereas the graph of $g(x)$ has the reverse change at point K. Consequently, instead of characterizing an inflection point as a point where the first derivative reaches an extreme value, we may alternatively characterize it as a point where the function undergoes a change in curvature or a change in the sign of its second derivative.

An Application

The two curves in Fig. 9.5 exemplify the graphs of quadratic functions, which may be expressed generally in the form

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

From our discussion of the second derivative, we can now derive a convenient way of determining whether a given quadratic function will have a strictly convex (U-shaped) or a strictly concave (inverse U-shaped) graph.

Since the second derivative of the quadratic function cited is $d^2y/dx^2 = 2a$, this derivative will always have the same algebraic sign as the coefficient a. Recalling that a positive second derivative implies a strictly convex curve, we can infer that a positive coefficient a in the above quadratic function gives rise to a U-shaped graph. In contrast, a negative coefficient a leads to a strictly concave curve, shaped like an inverted U.

As intimated at the end of Sec. 9.2, the relative extremum of this function will also prove to be its absolute extremum, because in a quadratic function there can be found only a single valley or peak, evident in a U or inverted U, respectively.

EXERCISE 9.3

1 Find the second and third derivatives of the following functions:

(*a*) $ax^2 + bx + c$ (*c*) $\dfrac{2x}{1 - x} \quad (x \neq 1)$

(*b*) $6x^4 - 3x - 4$ (*d*) $\dfrac{1 + x}{1 - x} \quad (x \neq 1)$

2 Which of the following quadratic functions are strictly convex?
(a) $y = 9x^2 - 4x + 2$ (c) $u = 9 - x^2$
(b) $w = -3x^2 + 39$ (d) $v = 8 - 3x + x^2$

3 Draw (a) a concave curve which is *not* strictly concave, and (b) a curve which qualifies simultaneously as a concave curve and a convex curve.

4 Given the function $y = a - \dfrac{b}{c + x}$ $(a, b, c > 0;\ x \geq 0)$, determine the general shape of its graph by examining (a) its first and second derivatives, (b) its vertical intercept, and (c) the limit of y as x tends to infinity. If this function is to be used as a consumption function, how should the parameters be restricted in order to make it economically sensible?

5 Draw the graph of a function $f(x)$ such that $f'(x) = 0$, and the graph of a function $g(x)$ such that $g'(3) = 0$. Summarize in one sentence the essential difference between $f(x)$ and $g(x)$ in terms of the concept of stationary point.

9.4 SECOND-DERIVATIVE TEST

Returning to the pair of extreme points B and E in Fig. 9.5 and remembering the newly established relationship between the second derivative and the curvature of a curve, we should be able to see the validity of the following criterion for a relative extremum:

Second-derivative test for relative extremum If the first derivative of a function f at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the second-derivative value at x_0 is $f''(x_0) < 0$.
b. A relative *minimum* if the second-derivative value at x_0 is $f''(x_0) > 0$.

This test is in general more convenient to use than the first-derivative test, because it does not require us to check the derivative sign to both the left and the right of x_0. But it has the drawback that no unequivocal conclusion can be drawn in the event that $f''(x_0) = 0$. For then the stationary value $f(x_0)$ can be *either* a relative maximum, *or* a relative minimum, *or* even an inflectional value.* When the situation of $f''(x_0) = 0$ is encountered, we must either revert to the first-derivative test, or resort to another test, to be developed in Sec. 9.6, that involves

* To see that an inflection point is possible when $f''(x_0) = 0$, let us refer back to Fig. 9.3a and 9.3a'. Point J in the upper diagram is an inflection point, with $x = j$ as its critical value. Since the $f'(x)$ curve in the lower diagram attains a minimum at $x = j$, the slope of $f'(x)$ [i.e., $f''(x)$] must be zero at the critical value $x = j$. Thus point J illustrates an inflection point occurring when $f''(x_0) = 0$.

To see that a relative extremum is also consistent with $f''(x_0) = 0$, consider the function $y = x^4$. This function plots as a U-shaped curve and has a minimum, $y = 0$, attained at the critical value $x = 0$. Since the second derivative of this function is $f''(x) = 12x^2$, we again obtain a zero value for this derivative at the critical value $x = 0$. Thus this function illustrates a relative extremum occurring when $f''(x_0) = 0$.

the third or even higher derivatives. For most problems in economics, however, the second-derivative test should prove to be adequate for determining a relative maximum or minimum.

Example 1 Find the relative extremum of the function

$$y = f(x) = 4x^2 - x$$

The first and second derivatives are

$$f'(x) = 8x - 1 \qquad \text{and} \qquad f''(x) = 8$$

Setting $f'(x)$ equal to zero and solving the resulting equation, we find the (only) critical value to be $\bar{x} = \frac{1}{8}$, which yields the (only) stationary value $f(\frac{1}{8}) = -\frac{1}{16}$. Because the second derivative is positive (in this case it is indeed positive for any value of x), the extremum is established as a minimum. Indeed, since the given function plots as a U-shaped curve, the relative minimum is also the absolute minimum.

Example 2 Find the relative extrema of the function

$$y = g(x) = x^3 - 3x^2 + 2$$

The first two derivatives of this function are

$$g'(x) = 3x^2 - 6x \qquad \text{and} \qquad g''(x) = 6x - 6$$

Setting $g'(x)$ equal to zero and solving the resulting quadratic equation, $3x^2 - 6x = 0$, we obtain the critical values $\bar{x}_1 = 0$ and $\bar{x}_2 = 2$, which in turn yield the two stationary values:

$$g(0) = 2 \qquad [\text{a maximum because } g''(0) = -6 < 0]$$

$$g(2) = -2 \qquad [\text{a minimum because } g''(2) = 6 > 0]$$

Necessary versus Sufficient Conditions

As was the case with the first-derivative test, the zero-slope condition $f'(x) = 0$ plays the role of a *necessary* condition in the second-derivative test. Since this condition is based on the first-order derivative, it is often referred to as the *first-order condition*. Once we find the first-order condition satisfied at $x = x_0$, the negative (positive) sign of $f''(x_0)$ is *sufficient* to establish the stationary value in question as a relative maximum (minimum). These sufficient conditions, which are based on the second-order derivative, are often referred to as *second-order conditions*.

It bears repeating that the first-order condition is *necessary*, but *not sufficient*, for a relative maximum or minimum. (Remember inflection points?) In sharp contrast, while the second-order condition that $f''(x)$ be negative (positive) at the critical value x_0 is *sufficient* for a relative maximum (minimum), it is *not necessary*. [Remember the relative extremum that occurs when $f''(x_0) = 0$?] For this reason, one should carefully guard against the following line of argument: "Since the stationary value $f(x_0)$ is already known to be a minimum, we must have $f''(x_0) > 0$." The reasoning here is faulty because it incorrectly treats the positive sign of $f''(x_0)$ as a necessary condition for $f(x_0)$ to be a minimum.

This is not to say that second-order derivatives can never be used in stating *necessary* conditions for relative extrema. Indeed they can. But care must then be taken to allow for the fact that a relative maximum (minimum) can occur not only when $f''(x_0)$ is negative (positive), but also when $f''(x_0)$ is zero. Consequently, *second-order necessary conditions* must be couched in terms of weak inequalities: for a stationary value $f(x_0)$ to be a relative $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right\}$, it is necessary that $f''(x_0)\left\{\begin{matrix}\leq\\ \geq\end{matrix}\right\}0$.

Conditions for Profit Maximization

We shall now present some economic examples of extreme-value problems, i.e., problems of optimization.

One of the first things that a student of economics learns is that, in order to maximize profit, a firm must equate marginal cost and marginal revenue. Let us show the mathematical derivation of this condition. To keep the analysis on a general level, we shall work with the total-revenue function $R = R(Q)$ and total-cost function $C = C(Q)$, both of which are functions of a single variable Q. From these it follows that a profit function (the objective function) may also be formulated in terms of Q (the choice variable):

$$\pi = \pi(Q) = R(Q) - C(Q) \tag{9.1}$$

To find the profit-maximizing output level, we must satisfy the first-order necessary condition for a maximum: $d\pi/dQ = 0$. Accordingly, let us differentiate (9.1) with respect to Q and set the resulting derivative equal to zero. The result is

$$\begin{aligned}\frac{d\pi}{dQ} &\equiv \pi'(Q) = R'(Q) - C'(Q)\\ &= 0 \qquad \text{iff} \qquad R'(Q) = C'(Q)\end{aligned} \tag{9.2}$$

Thus the *optimum* output (*equilibrium* output) $\bar{Q}$ must satisfy the equation $R'(\bar{Q}) = C'(\bar{Q})$, or MR = MC. This condition constitutes the first-order condition for profit maximization.

However, the first-order condition may lead to a minimum rather than a maximum; thus we must check the second-order condition next. We can obtain the second derivative by differentiating the first derivative in (9.2) with respect to Q:

$$\begin{aligned}\frac{d^2\pi}{dQ^2} &\equiv \pi''(Q) = R''(Q) - C''(Q)\\ &< 0 \qquad \text{iff} \qquad R''(Q) < C''(Q)\end{aligned}$$

For an output level $\bar{Q}$ such that $R'(\bar{Q}) = C'(\bar{Q})$, the satisfaction of the second-order condition $R''(\bar{Q}) < C''(\bar{Q})$ is sufficient to establish it as a profit-maximizing output. Economically, this would mean that, if the rate of change of MR is less than the rate of change of MC at the output where MC = MR, then that output will maximize profit.

These conditions are illustrated in Fig. 9.6. In diagram *a* we have drawn a total-revenue and a total-cost curve, which are seen to intersect twice, at output levels of Q_2 and Q_4. In the open interval (Q_2, Q_4), total revenue R exceeds total cost C, and thus π is positive. But in the intervals $[0, Q_2)$ and $(Q_4, Q_5]$, where Q_5 represents the upper limit of the firm's productive capacity, π is negative. This fact is reflected in diagram *b*, where the profit curve—obtained by plotting the vertical distance between the R and C curves for each level of output—lies above the horizontal axis only in the interval (Q_2, Q_4).

When we set $d\pi/dQ = 0$, in line with the first-order condition, it is our intention to locate the peak point K on the profit curve, at output Q_3, where the slope of the curve is zero. However, the relative-minimum point M (output Q_1) will also offer itself as a candidate, because it, too, meets the zero-slope requirement. We shall later resort to the second-order condition to eliminate the "wrong" kind of extremum.

The first-order condition $d\pi/dQ = 0$ is equivalent to the condition $R'(Q) = C'(Q)$. In Fig. 9.6*a*, the output level Q_3 satisfies this, because the R and C curves do have the same slope at Q_3 (the tangent lines drawn to the two curves at H and J are parallel to each other). The same is true for output Q_1. Since the equality of the slopes of R and C means the equality of MR and MC, outputs Q_3 and Q_1 must obviously be where the MR and MC curves intersect, as illustrated in Fig. 9.6*c*.

How does the second-order condition enter into the picture? Let us first look at Fig. 9.6*b*. At point K, the second derivative of the π function will (barring the exceptional zero-value case) have a negative value, $\pi''(Q_3) < 0$, because the curve is inverse U-shaped around K; this means that Q_3 will maximize profit. At point M, on the other hand, we would expect that $\pi''(Q_1) > 0$; thus Q_1 provides a relative minimum for π instead. The second-order sufficient condition for a maximum can, of course, be stated alternatively as $R''(Q) < C''(Q)$, that is, that the slope of the MR curve be less than the slope of the MC curve. From Fig. 9.6*c*, it is immediately apparent that output Q_3 satisfies this condition, since the slope of MR is negative while that of MC is positive at point L. But output Q_1 violates this condition because both MC and MR have negative slopes, and that of MR is *numerically smaller* than that of MC at point N, which implies that $R''(Q_1)$ is *greater* than $C''(Q_1)$ instead. In fact, therefore, output Q_1 also violates the second-order *necessary* condition for a relative maximum, but satisfies the second-order *sufficient* condition for a relative minimum.

Example 3 Let the $R(Q)$ and $C(Q)$ functions be

$$R(Q) = 1200Q - 2Q^2$$

$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

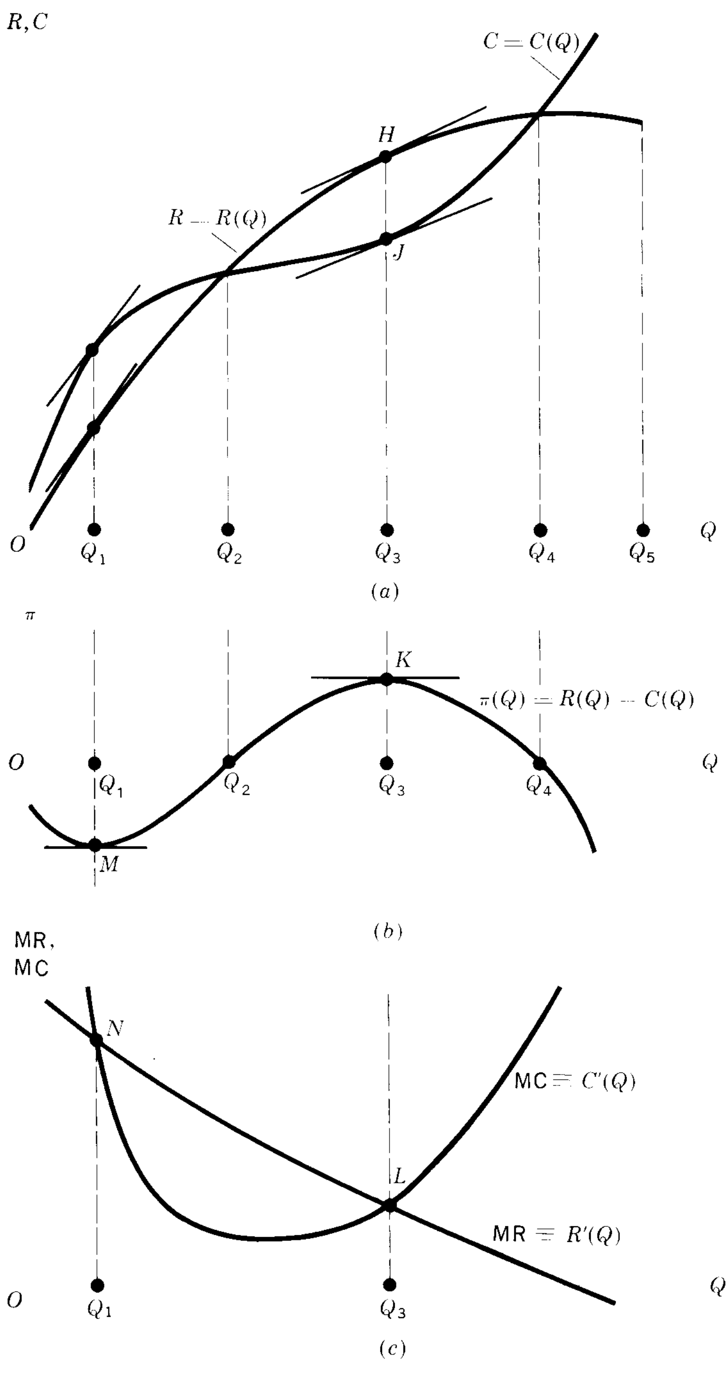
R, C
C = C(Q)
H
R = R(Q)
J
O
Q
Q1
Q2
Q3
Q4
Q5
(a)
π
K
π(Q) = R(Q) − C(Q)
O
Q
Q1
Q2
Q3
Q4
M
(b)
MR,
MC
N
MC ≡ C′(Q)
L
MR ≡ R′(Q)
O
Q
Q1
Q3
(c)

Figure 9.6

Then the profit function is

$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

where R, C, and π are all in dollar units and Q is in units of (say) tons per week. This profit function has two critical values, $Q = 3$ and $Q = 36.5$, because

$$\frac{d\pi}{dQ} = -3Q^2 + 118.5Q - 328.5 = 0 \qquad \text{when } Q = \begin{cases} 3 \\ 36.5 \end{cases}$$

But since the second derivative is

$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \qquad \begin{cases} > 0 & \text{when } Q = 3 \\ < 0 & \text{when } Q = 36.5 \end{cases}$$

the profit-maximizing output is $\bar{Q} = 36.5$ (tons per week). (The other output minimizes profit.) By substituting $\bar{Q}$ into the profit function, we can find the maximized profit to be $\bar{\pi} = \pi(36.5) = 16{,}318.44$ (dollars per week).

As an alternative approach to the above, we can first find the MR and MC functions and then equate the two, i.e., find their intersection. Since

$$R'(Q) = 1200 - 4Q$$

$$C'(Q) = 3Q^2 - 122.5Q + 1528.5$$

equating the two functions will result in a quadratic equation identical with $d\pi/dQ = 0$ which has yielded the two critical values of Q cited above.

Coefficients of a Cubic Total-Cost Function

In Example 3 above, a cubic function is used to represent the total-cost function. The traditional total-cost curve $C = C(Q)$, as illustrated in Fig. 9.6a, is supposed to contain two wiggles that form a concave segment (decreasing marginal cost) and a subsequent convex segment (increasing marginal cost). Since the graph of a cubic function always contains exactly two wiggles, as illustrated in Fig. 9.4, it should suit that role well. However, Fig. 9.4 immediately alerts us to a problem: the cubic function can possibly produce a downward-sloping segment in its graph, whereas the total-cost function, to make economic sense, should be upward-sloping everywhere (a larger output always entails a higher total cost). If we wish to use a cubic total-cost function such as

$$C = C(Q) = aQ^3 + bQ^2 + cQ + d \tag{9.3}$$

therefore, it is essential to place appropriate restrictions on the parameters so as to prevent the C curve from ever bending downward.

An equivalent way of stating this requirement is that the MC function should be positive throughout, and this can be ensured only if the *absolute minimum* of the MC function turns out to be positive. Differentiating (9.3) with respect to Q, we obtain the MC function

$$\text{MC} = C'(Q) = 3aQ^2 + 2bQ + c \tag{9.4}$$

which, because it is a quadratic, plots as a parabola as in Fig. 9.6*c*. In order for the MC curve to stay positive (above the horizontal axis) everywhere, it is necessary that the parabola be U-shaped (otherwise, with an inverse U, the curve is bound to extend itself into the second quadrant). Hence the coefficient of the Q^2 term in (9.4) has to be positive; i.e., we must impose the restriction $a > 0$. This restriction, however, is by no means sufficient, because the minimum value of a U-shaped MC curve—call it MC_{min} (a relative minimum which also happens to be an absolute minimum)—may still occur below the horizontal axis. Thus we must next find MC_{min} and ascertain the parameter restrictions that would make it positive.

According to our knowledge of relative extremum, the minimum of MC will occur where

$$\frac{d}{dQ}\text{MC} = 6aQ + 2b = 0$$

The output level that satisfies this first-order condition is

$$Q^* = \frac{-2b}{6a} = \frac{-b}{3a}$$

This minimizes (rather than maximizes) MC because the second derivative $d^2(\text{MC})/dQ^2 = 6a$ is assuredly positive in view of the restriction $a > 0$. The knowledge of Q^* now enables us to calculate MC_{min}, but we may first infer the sign of coefficient b from it. Inasmuch as negative output levels are ruled out, we see that b can never be positive (given $a > 0$). Moreover, since the law of diminishing returns is assumed to set in at a positive output level (that is, MC is assumed to have an initial declining segment), Q^* should be positive (rather than zero). Consequently, we must impose the restriction $b < 0$.

It is a simple matter now to substitute the MC-minimizing output Q^* into (9.4) to find that

$$\text{MC}_{min} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\frac{-b}{3a} + c = \frac{3ac - b^2}{3a}$$

Thus, to guarantee the positivity of MC_{min}, we must impose the restriction* $b^2 < 3ac$. This last restriction, we may add, in effect also implies the restriction $c > 0$. (Why?)

* This restriction may also be obtained by the method of *completing the square*. The MC function can be successively transformed as follows:

$$\begin{aligned}\text{MC} &= 3aQ^2 + 2bQ + c \\ &= \left(3aQ^2 + 2bQ + \frac{b^2}{3a}\right) - \frac{b^2}{3a} + c \\ &= \left(\sqrt{3a}\,Q + \sqrt{\frac{b^2}{3a}}\right)^2 + \frac{-b^2 + 3ac}{3a}\end{aligned}$$

Since the squared expression can possibly be zero, the positivity of MC will be ensured—on the knowledge that $a > 0$—only if $b^2 < 3ac$.

The above discussion has involved the three parameters a, b, and c. What about the other parameter, d? The answer is that there is need for a restriction on d also, but that has nothing to do with the problem of keeping the MC positive. If we let $Q = 0$ in (9.3), we find that $C(0) = d$. The role of d is thus to determine the vertical intercept of the C curve only, with no bearing on its slope. Since the economic meaning of d is the fixed cost of a firm, the appropriate restriction (in the short-run context) would be $d > 0$.

In sum, the coefficients of the total-cost function (9.3) should be restricted as follows (assuming the short-run context):

$$(9.5) \qquad a, c, d > 0 \qquad b < 0 \qquad b^2 < 3ac$$

As you can readily verify, the $C(Q)$ function in Example 3 does satisfy (9.5).

Upward-Sloping Marginal-Revenue Curve

The marginal-revenue curve in Fig. 9.6c is shown to be downward-sloping throughout. This, of course, is how the MR curve is traditionally drawn for a firm under imperfect competition. However, the possibility of the MR curve being partially, or even wholly, upward-sloping can by no means be ruled out a priori.*

Given an average-revenue function AR $= f(Q)$, the marginal-revenue function can be expressed by

$$\text{MR} = f(Q) + Qf'(Q) \qquad [\text{from (7.7)}]$$

The slope of the MR curve can thus be ascertained from the derivative

$$\frac{d}{dQ}\text{MR} = f'(Q) + f'(Q) + Qf''(Q) = 2f'(Q) + Qf''(Q)$$

As long as the AR curve is downward-sloping (as it would be under imperfect competition), the $2f'(Q)$ term is assuredly negative. But the $Qf''(Q)$ term can be either negative, zero, or positive, depending on the sign of the second derivative of the AR function, i.e., depending on whether the AR curve is strictly concave, linear, or strictly convex. If the AR curve is strictly convex either in its entirety (as illustrated in Fig. 7.2) or along a specific segment, the possibility will exist that the (positive) $Qf''(Q)$ term may dominate the (negative) $2f'(Q)$ term, thereby causing the MR curve to be wholly or partially upward-sloping.

Example 4 Let the average-revenue function be

$$\text{AR} = f(Q) = 8000 - 23Q + 1.1Q^2 - 0.018Q^3$$

As can be verified (see Exercise 9.4-7), this function gives rise to a downward-sloping AR curve, as is appropriate for a firm under imperfect competition. Since

$$\text{MR} = f(Q) + Qf'(Q) = 8000 - 46Q + 3.3Q^2 - 0.072Q^3$$

* This point is emphatically brought out in John P. Formby, Stephen Layson, and W. James Smith, "The Law of Demand, Positive Sloping Marginal Revenue, and Multiple Profit Equilibria," *Economic Inquiry*, April 1982, pp. 303–311.

it follows that the slope of MR is

$$\frac{d}{dQ}\text{MR} = -46 + 6.6Q - 0.216Q^2$$

Because this is a quadratic function and since the coefficient of Q^2 is negative, $d\text{MR}/dQ$ must plot as an inverse-U-shaped curve against Q, such as shown in Fig. 9.5*a*. If a segment of this curve happens to lie above the horizontal axis, therefore, the slope of MR will take positive values.

Setting $d\text{MR}/dQ = 0$, and applying the quadratic formula, we find the two zeros of the quadratic function to be $Q_1 = 10.76$ and $Q_2 = 19.79$ (approximately). This means that, for values of Q in the open interval (Q_1, Q_2), the $d\text{MR}/dQ$ curve does lie above the horizontal axis. Thus the marginal-revenue curve indeed is positively sloped for output levels between Q_1 and Q_2.

The presence of a positively sloped segment on the MR curve has interesting implications. With more bends in its configuration, such an MR curve may produce more than one intersection with the MC curve satisfying the second-order sufficient condition for profit maximization. While all such intersections constitute local optima, however, only one of them is the global optimum that the firm is seeking.

EXERCISE 9.4

1 Find the relative maxima and minima of y by the second-derivative test:

(*a*) $y = -2x^2 + 8x + 25$ (*c*) $y = \frac{1}{3}x^3 - 3x^2 + 5x + 3$

(*b*) $y = x^3 + 6x^2 + 7$ (*d*) $y = \frac{2x}{1 - 2x} \quad \left(x \neq \frac{1}{2}\right)$

2 Mr. Greenthumb wishes to mark out a rectangular flower bed along the side wall of his house. The other three sides are to be marked by wire netting, of which he has only 32 ft available. What are the length L and width W of the rectangle that would give him the largest possible planting area? How do you make sure that your answer gives the largest, not the smallest area?

3 A firm has the following total-cost and demand functions:

$$C = \tfrac{1}{3}Q^3 - 7Q^2 + 111Q + 50$$

$$Q = 100 - P$$

(*a*) Does the total-cost function satisfy the coefficient restrictions of (9.5)?
(*b*) Write out the total-revenue function R in terms of Q.
(*c*) Formulate the total-profit function π in terms of Q.
(*d*) Find the profit-maximizing level of output $\bar{Q}$.
(*e*) What is the maximum profit?

4 If coefficient b in (9.3) were to take a zero value, what would happen to the marginal-cost and total-cost curves?

5 A quadratic profit function $\pi(Q) = hQ^2 + jQ + k$ is to be used to reflect the following assumptions:

(a) If nothing is produced, the profit will be negative (because of fixed costs).

(b) The profit function is strictly concave.

(c) The maximum profit occurs at a positive output level $\bar{Q}$.

What parameter restrictions are called for?

6 A purely competitive firm has a single variable input L (labor), with the wage rate W per period. Its fixed inputs cost the firm a total of F dollars per period. The price of the product is P_0.

(a) Write the production function, revenue function, cost function, and profit function of the firm.

(b) What is the first-order condition for profit maximization? Interpret the condition economically.

(c) What economic circumstances would ensure that profit is maximized rather than minimized?

7 Use the following procedure to verify that the AR curve in Example 4 is negatively sloped:

(a) Denote the slope of AR by S. Write an expression for S.

(b) Find the maximum value of S, S_{max}, by using the second-derivative test.

(c) Then deduce from the value of S_{max} that the AR curve is negatively sloped.

9.5 DIGRESSION ON MACLAURIN AND TAYLOR SERIES

The time has now come for us to develop a test for relative extrema that can apply even when the second derivative turns out to have a zero value at the stationary point. Before we can do that, however, it will first be necessary to discuss the so-called "expansion" of a function $y = f(x)$ into what are known, respectively, as a *Maclaurin series* (expansion around the point $x = 0$) and a *Taylor series* (expansion around any point $x = x_0$).

To *expand* a function $y = f(x)$ around a point x_0 means, in the present context, to transform that function into a *polynomial* form, in which the coefficients of the various terms are expressed in terms of the derivative values $f'(x_0)$, $f''(x_0)$, etc.—all evaluated at the point of expansion x_0. In the Maclaurin series, these will be evaluated at $x = 0$; thus we have $f'(0)$, $f''(0)$, etc., in the coefficients. The result of expansion may be referred to as a *power series* because, being a polynomial, it consists of a sum of power functions.

Maclaurin Series of a Polynomial Function

Let us consider first the expansion of a *polynomial* function of the nth degree,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n \tag{9.6}$$

Since this involves the transformation of one polynomial into another, it may seem a sterile and purposeless exercise, but actually it will serve to shed much light on the whole idea of expansion.